Topological and Metric Analysis of Heteroclinic Crisis in Laser Chaos

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(Received 30 October 1991)

The power-law behavior of the average time between intermittent bursts in the NMR-laser dynamics near a heteroclinic tangency crisis is investigated. Using symbolic-dynamical techniques, new crisis-induced sequences are identified in the strange attractors reconstructed from both experimental and simulated data. Our approach provides a precise criterion for the onset of the attractor widening due to the collision of the stable and the unstable manifold belonging to different unstable periodic orbits. The results show the predictability power of our laser model.

PACS numbers: 05.45.+b, 42.65.−k, 76.60.−k

Sudden changes occurring in strange attractors upon variation of control parameters have been so far distinguished into three different classes [1]. While the first and second types of transitions lead either to a destruction or to a widening of the chaotic attractor, respectively, the third one corresponds to a merging of two or more strange sets. These transitions are termed "crises" and are accompanied by a characteristic temporal scaling of trajectories above the critical point. We concentrate on the second case, in which the dynamics presents an intermittent bursting out of the phase-space region within which the attractor was confined before the crisis (core attractor). The average time τ between the bursts of this so-called "crisis-induced intermittency" [1] obeys the power law $\tau(p) \propto |p-p_c|^{-\gamma}$ in the vicinity of the critical value p_c of the control parameter p, which marks the onset of the crisis. The critical scaling exponent γ has been predicted to lie in the interval $[\frac{1}{2}, \frac{3}{2}]$ (where the extrema correspond to the strongly dissipative and conservative limits, respectively) [1]. At $p=p_c$ the attractor (the closure of the unstable manifold of some periodic orbit A) collides with the stable manifold of an unstable periodic orbit B. If A and B are just the same orbit, the crisis is called homoclinic, otherwise, it is called heteroclinic. The exponent γ for the heteroclinic crisis is given by $\gamma=\frac{1}{2}+\lambda_1/|\lambda_2|$, where λ_1 and λ_2 are the expanding and contracting Lyapunov exponents of the mediating periodic orbit, respectively [1].

So far, only a homoclinic tangency crisis has been observed experimentally in an externally driven magnetoelastic ribbon [2,3]. In the following, we present first experimental evidence of crisis-induced intermittency via a heteroclinic tangency, taking advantage of a parametrically modulated NMR-laser system. While such a type of crisis has already appeared in many previous investigations on laser chaos near the critical transition point p_c [4], the results of the present Letter quantitatively characterize the temporal scaling behavior of the orbits. By analyzing the time series obtained from the experiment and from the extended Bloch-type laser model [4,5] with symbolic-dynamical techniques, the critical scaling exponent has been estimated and confirmed through independent calculation of the Lyapunov exponents. The accuracy of the method stems from its ability to resolve neighborhoods of high-order unstable periodic orbits.

The NMR-laser [4] activity is provided by pumped nuclear ^{27}Al spins in a ruby crystal, placed in a static magnetic field $\mathbf{B}_0$ of magnitude 1.1 T at a temperature of 4.2 K. The total nuclear magnetization $\mathbf{M}=(M_x,M_y,M_z)$ precesses with the NMR frequency $\nu_a=12.3$ MHz. The spin population inversion is obtained by means of a microwave pump (dynamical nuclear polarization), and the laser action by enclosing the active medium in a cavity (in our case, an LC circuit tuned to ν_a for single-mode selection). This provides the feedback radiation field $\mathbf{B}$ (proportional to the current in the circuit) necessary for coherent spin-flip behavior. Furthermore, the cavity is forced to operate with a sinusoidally varying quality factor $Q(t)=Q_0(1+p\cos 2\pi\nu_m t)$, where p (the control parameter) is the modulation amplitude and ν_m the modulation frequency. The laser output corresponds to the voltage across the LC circuit, proportional to the transverse nuclear magnetization amplitude $M_t=(M_x^2+M_y^2)^{1/2}$.

An adequate description of the laser dynamics in the rotating frame is given by the extended Bloch-type model [4,5]

$$\begin{aligned} \dot{x} &= \sigma[y - x/f(t')] , \\ \dot{y} &= -y(1+ay)+rx-xz , \qquad (1) \\ \dot{z} &= -bz+xy , \end{aligned}$$

where the overdots represent the derivative with respect to a rescaled adimensional time t' and $x \propto B_t$, $y \propto M_t$, $z \propto M_z - M_e$. B_t denotes the rotating field amplitude and M_e the pump magnetization. The proportionality factors, as well as the parameters $\sigma=4.875$, $r=1.807 \propto M_e$, and $b=2\times10^{-4}$, depend on various physical constants. The function $f(t')=1+p\cos(2\pi\nu_m t')$ describes the parameter modulation with frequency ν_m.

In order to describe our results in a more general framework, we use methods of the symbolic-dynamics theory [6]. This approach includes all the relevant metric and topological features of the system under investigation. Once the attractor has been reconstructed by embedding the time series in a suitable auxiliary space, all unstable periodic orbits up to some order n_{max} are localized. A Poincaré section is then chosen in such a way that the same number n of intersections is obtained for all period-T orbits $(T = n/\nu_m)$. Finally, a generating partition is approximated by attributing different symbolic sequences to each unstable periodic point (and to their neighborhoods).

We have collected 14 different experimental time series $\{\xi_1, \xi_2, \ldots, \xi_N\}$ by sampling the laser output with a frequency $\nu_s = 4\nu_m$. They consist of $N = 10^5$ 12-bit integers. The modulation frequency, kept constant at $\nu_m = 120$ Hz, corresponds to about twice the intrinsic relaxation frequency. The modulation amplitude was varied in the range between $p \approx 0.01800$ and $p \approx 0.01870$ (where the former value is slightly below the crisis point $p_c \approx 0.01802$ and the latter well above it). The experimental data have been reconstructed in an E-dimensional embedding space [7] by forming vectors $v_i = \{\xi_i, \xi_{i+1}, \ldots, \xi_{i+E-1}\}$. The unstable periodic orbits of the system have been located by looking at portions of reconstructed trajectories recurring in certain spherical regions of radius R within distinct time intervals (chosen to be integer multiples of the period of the external forcing term). The value of R gives the precision with which the unstable periodic orbits are shadowed. In order to minimize the relative error in the search, R is chosen to be proportional to the local density of points in phase space and to the square root of the embedding dimension E. Tests have shown consistent results for E between 6 and 16. For comparison, the unstable periodic orbits of the model were extracted by a straightforward application of the Newton method. Up to order 9, we found complete correspondence with the experimental results: Not only do all periodic points have the same symbolic representations, but also the shapes of the orbits are in close resemblance.

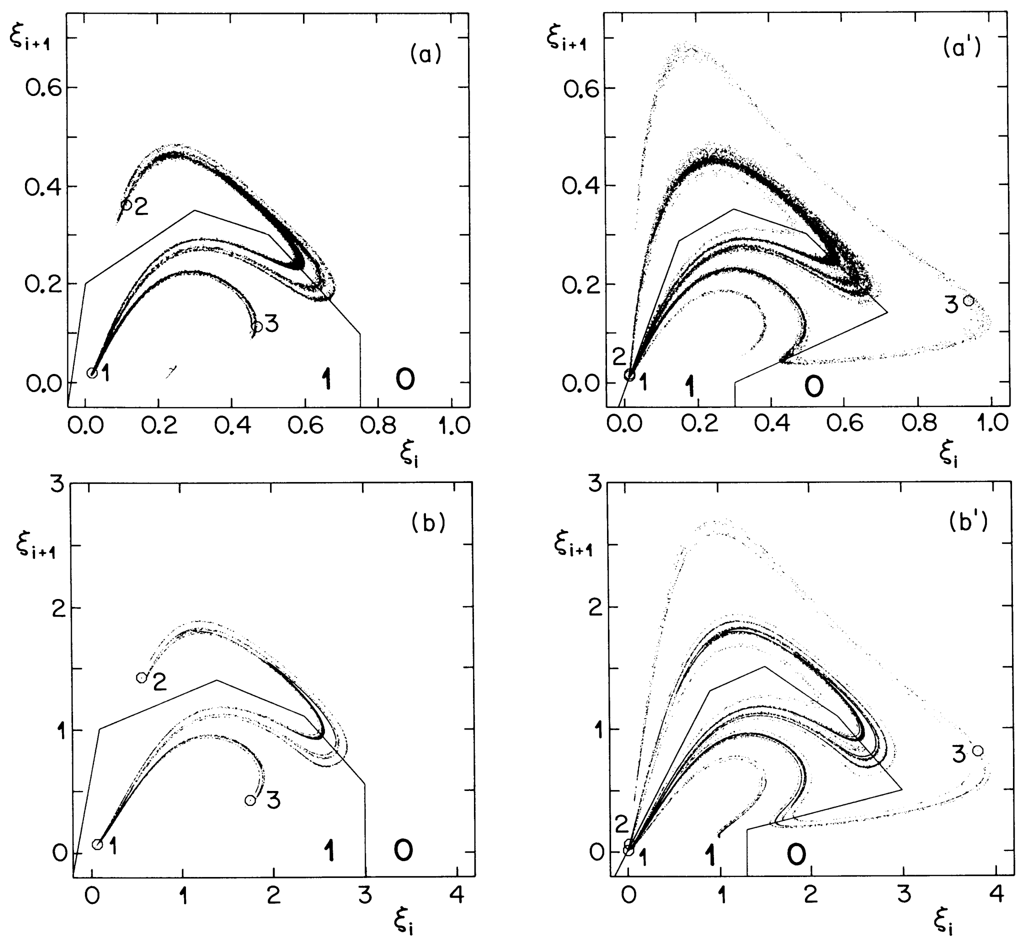

FIG. 1. Poincaré sections obtained from (a),(a') experiment and (b),(b') model for the parameters (a),(b) $p = 0.0180$ and (a'),(b') $p = 0.0185$. The intersection points (open circles) of the old unstable period-3 orbit A [(a) and (b)] and of the new period-3 orbit B [(a') and (b')] are numbered in order of occurrence in time (symbolic sequences 011 and 001, respectively). The solid curves indicate an approximation to a generating partition with elements 0 and 1. Upon magnification, the first two intersection points 1 and 2 in (a') and (b') can be distinguished as located on opposite sides of the partition.

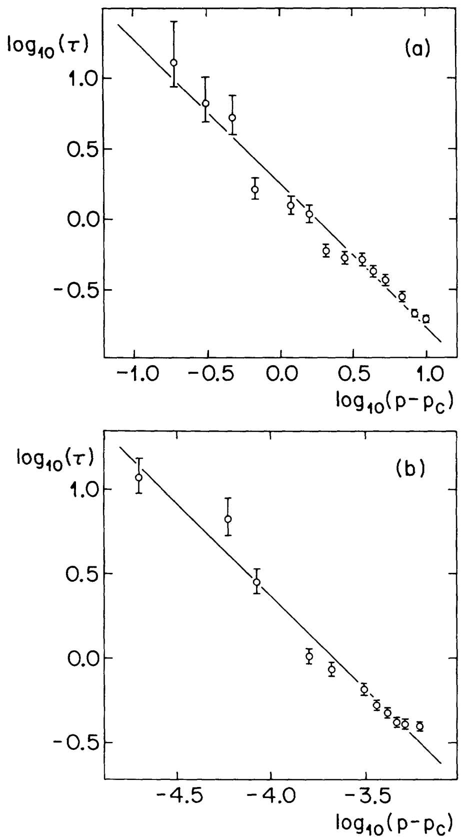

FIG. 2. Average time $\tau(p)$ between successive bursts as a function of $|p-p_c|$, obtained from (a) experiment and (b) model. The critical exponents (a) $\gamma_{\rm expt}=1.02\pm0.05$ and (b) $\gamma_{\rm mod}=1.10\pm0.05$ are determined from the slopes of the straight lines obtained by a least-squares fit in a doubly logarithmic scale.

In order to obtain a condensed description of the dynamics of the system, Poincaré sections have been constructed (Fig. 1). Successively, we have encoded the intersection points with all extracted unstable limit cycles by means of a binary partition $\mathcal{D}=\{\Delta_0,\Delta_1\}$, whose elements are labeled by the two symbols 0 and 1. This partition yields a good approximation to the generating one, since it is able to encode uniquely all periodic points up to order 9. Below the crisis, all points belonging to the subset Δ_0 are mapped to Δ_1 in one iteration. Therefore, the string 00 is forbidden and the most condensed description of the dynamics is given by a binary tree over the two primitive [8] words $w_1=1$ and $w_2=01$. After the onset of the crisis, the string 00 is no longer forbidden and, hence, the additional primitive word $w_3=001$ needs to be introduced. The attractor widening is associated with the appearance of the new unstable period-3 orbit B with label 001 [Figs. 1(a′) and 1(b′)], the "old" orbit A having label 011 [Figs. 1(a) and 1(b)]. The symbolic signal consists of combinations of w_1, w_2, and w_3. In Fig. 1, the intersection points of the two orbits are shown, numbered in order of appearance. Experiment and model are clearly in accordance.

Knowledge of the symbolic dynamics provides a precise criterion for the detection of the transition. If sequence 00 occurs, the second 0 is part of the new branch; i.e., the crisis has taken place. Hence, all points contained in element 00 in phase space are mapped onto such a region. They can be reached only from a small neighborhood of the point marked by a 1 in Figs. 1(a′) and 1(b′). Accordingly, we have estimated the average time τ spent in the old region and determined the critical exponent γ from the slopes of the curves $\log\tau(p)$ vs $\log|p-p_c|$, displayed in Fig. 2. The values obtained from experiment and model are $\gamma_{\rm expt}=1.02\pm0.05$ and $\gamma_{\rm mod}=1.10\pm0.05$, respectively. An independent determination of γ has been carried out by evaluating the eigenvalues of a linearized map around the A orbit, for different parameter values: We obtained $\gamma_{\rm lya}=1.15\pm0.10$, in agreement with the direct estimates.

To conclude, we have confirmed the theoretically predicted scaling behavior of crisis-induced intermittency for the heteroclinic tangency occurring in the dynamics of the Q-modulated NMR laser, for both experiment and model. In addition, we have proposed and demonstrated the feasibility of a detailed symbolic-dynamical description of strange attractors. Our results elucidate so far hidden aspects of attractor structure near crisis points.

We thank C. Grebogi and P. Talkner for stimulating discussions and the Swiss National Science Foundation for partial financial support.

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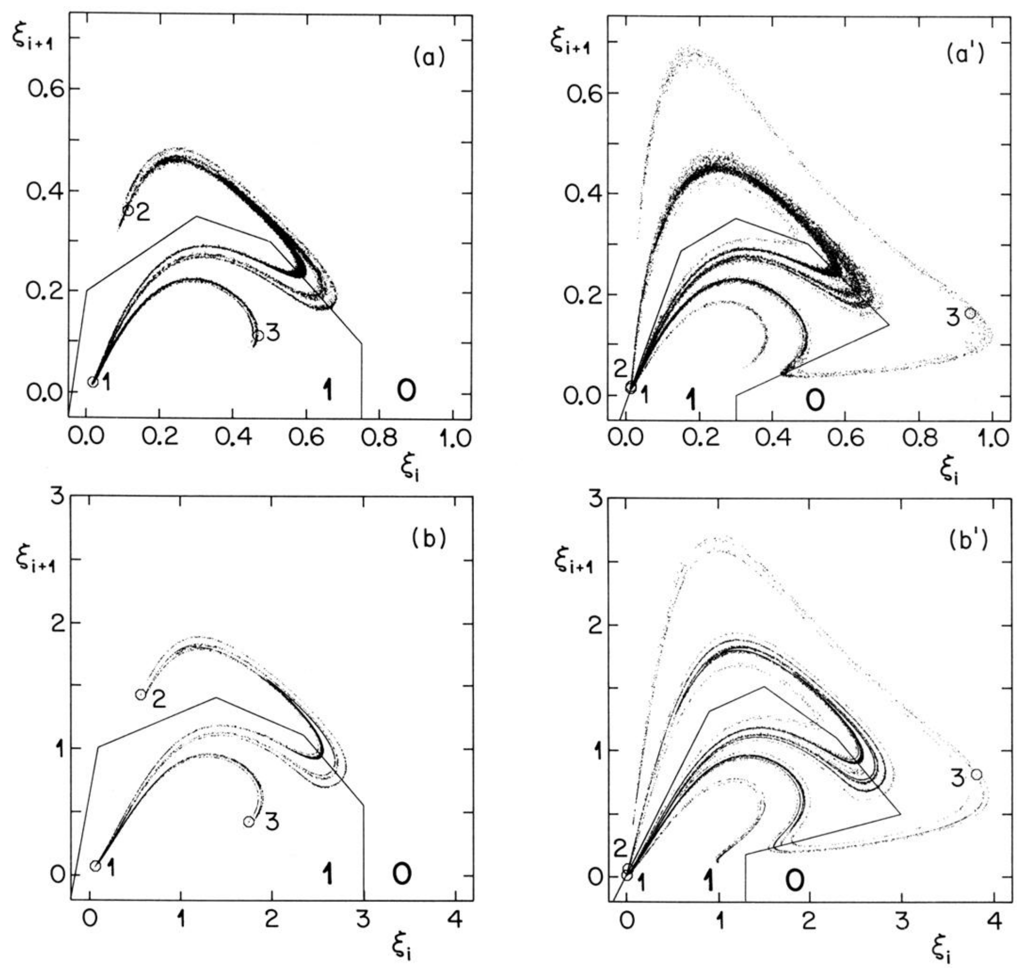

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