Phase Shifts in Stochastic Resonance

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The response of an overdamped bistable system to a weak periodic force in the presence of external noise is investigated using linear-response theory and by analog electronic experiment. It is shown, both theoretically and experimentally, that a phase lag ϕ exists between the force and the response, and that $|\phi|$ passes through a maximum when the noise intensity is tuned through the range where stochastic resonance occurs. These results clarify the interrelation between stochastic resonance and conventional resonance phenomena, and confirm that stochastic resonance can usefully be treated by linear-response theory.

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A characteristic feature of periodically driven underdamped oscillators, familiar to all physicists, is the existence of a phase lag ϕ between the periodic driving force $A\cos\Omega t$ and the periodic response of the system. As Ω is increased gradually from zero and swept through the resonant frequency Ω_0, ϕ decreases monotonically from zero, passes through $-90°$ when $\Omega = \Omega_0$, and approaches $-180°$ when $\Omega \gg \Omega_0$. It is natural to ask, therefore, whether or not similar phase shifts also occur in *stochastic* resonance. Stochastic resonance (SR) is a remarkable phenomenon [1,2] much in the news of late, in which a weak periodic signal, usually in a bistable system, can be optimally amplified by the addition of external noise of appropriate intensity. The answer to this question is less than obvious because there is still no commonly acknowledged understanding of the nature of SR and its place in the context of other resonant phenomena. Indeed, it has been pointed out [3] that SR in bistable systems does not qualify as a resonance phenomenon at all, in the normal sense of matching two frequencies. In addition to its intrinsic interest, the question is also of some fundamental importance: This is because the answer holds the key to a resolution of the controversy over whether [2] or not [4] SR can usefully be treated as a linear-response phenomenon within the context of standard statistical physics.

The presence or absence of phase shifts in SR is a conundrum of many years' standing. The first prediction of a phase shift seems to have been due to Nicolis [5] who concluded that $\phi = -\arctan(\Omega/W^{(0)})$, where $W^{(0)}$ is the sum of the transition rates out of each of the potential wells of the overdamped bistable system under consideration; similar results were also obtained subsequently for a two-state model by McNamara and Wiesenfeld [6] and by Presilla, Marchesoni, and Gammaitoni [7], although the latter authors expressed disbelief in their own conclusion. Jung [8] and Jung and Hanggi [9], on the other hand, reportedly [4] failed to find any evidence of phase shifts in their numerical investigations of SR; furthermore, Gammaitoni *et al.* claimed [4] that analog simulations [7,10] had ruled out any possibility of there being phase shifts in SR. Because the reality of such phase shifts follows automatically from the proposed [2] treatment of SR by linear-response theory (LRT), Gammaitoni *et al.* went so far as to suggest [4] that this was, in itself, a good reason to doubt the applicability of LRT to SR. After the present paper had been submitted, Gammaitoni *et al.* discovered some evidence for SR phase shifts in an electron-spin-resonance (ESR) experiment [11], but they apparently attempted no comparison with the LRT predictions [2], from which their results would appear to differ.

In this Letter we report the outcome of a new electronic experiment that has finally resolved this long-standing controversy. We demonstrate below, unequivocally and in considerable detail, both experimentally and theoretically that phase shifts do indeed occur in SR; they do not, however, take the form predicted by [5–7]. We treat the simplest nontrivial system: an overdamped Brownian particle moving in a symmetric bistable potential and, in addition, driven periodically,

$$\dot{q} + U'(q) = A\cos\Omega t + f(t), \quad U(q) = -\tfrac{1}{2}q^2 + \tfrac{1}{4}q^4 , \tag{1}$$

where $f(t)$ is the zero-mean Gaussian noise of intensity D,

$$\langle f(t)f(t')\rangle = 2D\delta(t-t') . \tag{2}$$

We discuss first the behavior predicted by LRT, and then compare the predictions with the results of the electronic experiment. The average value of the coordinate oscillates with a period $2\pi/\Omega$:

$$\langle q(t)\rangle = \sum_{n\geq 0} a(n)\cos[n\Omega t + \phi(n)] \, . \tag{3}$$

By symmetry arguments $[U(q) = U(-q)]$ the "partial" amplitudes $a(n)$ and phases $\phi(n)$ with even n are decoupled from those with odd n, and therefore the sum in (3) actually runs over odd $n = 2k + 1$ only, where $k \geq 0$ is a positive integer. For a weak periodic force $a(2k+1) \propto A^{2k+1}$ and

$$a(1) = A|\chi(\Omega)| \, ,$$
$$\phi \equiv \phi(1) = -\arctan[\mathrm{Im}\chi(\Omega)/\mathrm{Re}\chi(\Omega)] \quad (A \to 0) \, , \tag{4}$$

where $\chi(\Omega)$ is the susceptibility of the system [12]. It follows from (3) that in the spectral density of fluctuations (SDF),

$$Q(\omega) = \lim_{\tau\to\infty} (4\pi\tau)^{-1} \left| \int_{-\tau}^{\tau} dt\, q(t)\exp(i\omega t) \right|^2 , \tag{5}$$

there arise δ-shaped spikes at frequencies $\pm(2k+1)\Omega$ because, according to the principle of the decay of correlations, $\langle q(t)q(t')\rangle \to \langle q(t)\rangle\langle q(t')\rangle$ as $|t-t'| \to \infty$. The areas of the spikes are equal to $\frac{1}{4}a^2(2k+1)$. Following [13] the response of the system is often characterized by the ratio R of the area of the spike at frequency Ω to the value $Q^{(0)}(\Omega)$ of the SDF in the absence of periodic forcing. For weak forcing, (3)–(5) then yield a signal-to-noise ratio

$$R = \tfrac{1}{4}A^2|\chi(\Omega)|^2/Q^{(0)}(\Omega) \quad (A \to 0) \, . \tag{6}$$

The dependences of both the amplitude $a(1)$ and R on the noise intensity D are known to display bell-shaped peaks under certain conditions, and it is just this sort of behavior that constitutes SR. As is shown below, the phase $\phi \equiv \phi(1)$ as a function of D can display similar "resonant" behavior.

For the quasiequilibrium system under consideration, the quantities $\chi(\Omega)$ and $Q^{(0)}(\Omega)$ in (6) are interrelated [12] via the fluctuation-dissipation theorem,

$$\mathrm{Re}\chi(\omega) = \frac{2}{D}\mathrm{P}\int_0^\infty d\omega_1 Q^{(0)}(\omega_1)\omega_1^2(\omega_1^2 - \omega^2)^{-1} \, ,$$
$$\mathrm{Im}\chi(\omega) = (\pi\omega/D)Q^{(0)}(\omega) \, , \tag{7}$$

where P implies the Cauchy principal part. They can be calculated explicitly for relatively small noise intensities $D \ll \Delta U$, where ΔU is the depth of the (shallowest) potential well and $\Delta U = \frac{1}{4}$ for the model (1). In this range $Q^{(0)}(\omega)$ and $\chi(\omega)$ are given [14] by the sums of contributions from fluctuations about the equilibrium positions $q_n = (-1)^n$ $(n = 1,2)$ and from interwell transitions,

$$Q^{(0)}(\omega) = \sum_{n=1,2} w_n Q_n^{(0)}(\omega) + Q_{\mathrm{tr}}^{(0)}(\omega) \, ,$$
$$\chi(\omega) = \sum_{n=1,2} w_n \chi_n(\omega) + \chi_{\mathrm{tr}}(\omega) \, . \tag{8}$$

Here, w_n is the population of the nth stable state. The susceptibilities $\chi_n(\omega)$ and $\chi_{\mathrm{tr}}(\omega)$ are expressed in terms of $Q_n^{(0)}(\omega)$, $Q_{\mathrm{tr}}^{(0)}(\omega)$ by Eq. (7), and therefore only the spectral densities will be written down below in explicit form. For the model (1), $w_1 = w_2 = \frac{1}{2}$, $Q_1^{(0)}(\omega) = Q_2^{(0)}(\omega)$, and $\chi_1(\omega) = \chi_2(\omega)$. The spectral density for the intrawell vibrations $Q_n^{(0)}(\omega)$ can be obtained by expanding $U(q)$ about the equilibrium position q_n. Assuming that the nonlinear terms are small, and allowing for them by perturbation theory, one obtains

$$Q_n^{(0)}(\omega) \simeq L_n(\omega) - \pi L_n^2(\omega) \times \left[U_n^{(\mathrm{IV})} - 9U_n'''^2 U_n''(4U_n''^2 + \omega^2)^{-1}\right] ,$$
$$L_n(\omega) = (1/\pi)D(U_n''^2 + \omega^2)^{-1} \, , \tag{9}$$

where all derivatives are evaluated for $q = q_n = (-1)^n$. The contribution from interwell transitions [14,15] is

$$Q_{\mathrm{tr}}^{(0)}(\omega) = \frac{1}{\pi}w_1 w_2 \frac{(\langle q\rangle_1^{(0)} - \langle q\rangle_2^{(0)})^2 W^{(0)}}{W^{(0)2} + \omega^2} \quad (\omega \ll U_{1,2}'') \, ,$$
$$W^{(0)} \equiv W^{(0)}(D) = W_{12}^{(0)} + W_{21}^{(0)} \, , \tag{10}$$
$$\langle q\rangle_n^{(0)} = q_n - \tfrac{1}{2}DU_n'''(U_n'')^{-2} \, .$$

Here, $\langle q\rangle_n^{(0)}$ is the average value of the coordinate in the nth well neglecting interwell transitions and $W_{nm}^{(0)}$ is the probability of the transition $n \to m$ in the absence of periodic forcing [corrections [16] $\sim D/\Delta U$ to the Kramers expression for the transition probabilities are required in (10)]. In deriving (8) we have utilized the inequality $W^{(0)} \ll \Omega_r = U_{1,2}''$, implying that the transition probabilities are very much smaller than the relaxation rate of the system Ω_r.

To the lowest order in $D/\Delta U$, to zeroth order in Ω/Ω_r, but for arbitrary $\Omega/W^{(0)}$, the expressions for R (6) and for ϕ (4) resulting from (7)–(10) for the model (1) become

$$R = \frac{\pi A^2}{4D^2}\frac{\Omega_r^2 W^{(0)2} + \Omega^2 D^2}{\Omega_r^2 W^{(0)} + \Omega^2 D}, \quad \Omega, D \ll \Omega_r, \quad W^{(0)} \ll D \, ,$$
$$\phi = -\arctan\frac{(\Omega/\Omega_r)(\Omega_r^2 W^{(0)} + \Omega^2 D)}{\Omega_r W^{(0)2} + \Omega^2 D} \, , \tag{11}$$

where $\Omega_r = U_1'' = U_2'' = 2$. For very small D, where $W^{(0)} \ll (\Omega^2/\Omega_r^2)D$, it follows from (11) that $R \simeq \pi A^2/4D$, $\phi \simeq -\Omega/\Omega_r$. Thus for a fixed forcing frequency Ω, R decreases with increasing D, whereas ϕ remains small and nearly independent of D.

For larger values of D, it is straightforward to demonstrate that (11) implies that R will pass through a minimum and then increase again (i.e., onset of the SR phenomenon) until $D \sim \Delta U/2$, when (8)–(10) are no longer applicable (see the solid curve in the inset in Fig. 1). We would comment that the failure of the theory at large D occurs because the expressions used for $Q^{(0)}(\omega)$ then become poor approximations; it is not a failure of LRT as such. We would also comment that the LRT predictions of an initial fall in R with increasing D and of the existence of the minimum in R have been observed in many experiments, were remarked upon and treated in an "*ad hoc*" fashion by McNamara and Wiesenfeld [6], but have not been accounted for quantitatively by any theory of SR other than LRT [2].

The central interest of the present paper relates, however, to the behavior of ϕ. The LRT prediction for model (1), based on (8)–(10), is shown by the solid curve in Fig. 1. It is evident that $|\phi|$ rises steeply from its $D \to 0$ value, passes through a maximum at $D = D_{\max} < \Delta U$, and then decreases more slowly again with further increase of D. The analytic approximation (11) is readily shown to yield the same behavior. In contrast, the variation of ϕ with D as predicted by earlier theories [5–7] that accounted for the phase shift in a two-state approximation is shown by the dashed line.

These predictions have been tested by means of an electronic experiment, using a circuit of conventional design [17] to model (1); full details will be given elsewhere. It is immediately evident from the measurements (Fig. 1), first, that contrary to [4,8–10] large phase shifts do indeed occur as D is varied and, second, that the LRT prediction describes the data remarkably well. The form of the phase shifts differs from that predicted by the earlier (two-state) theories [5–7] . For (1), with the parameters used in the experiment, a maximum value of $-\phi = 68°$ is predicted by LRT to occur at $D_{\max} = 0.08$, which is to be compared with the experimental observation for $A = 0.04$ of $-\phi = (66 \pm 2)°$ at $D = 0.08 \pm 0.01$. In accordance with the LRT prediction, the decrease of $|\phi|$ for $D > D_{\max}$ is much more gradual than the rapid increase seen below $D_{\max}$. The measured ϕ is relatively insensitive to A.

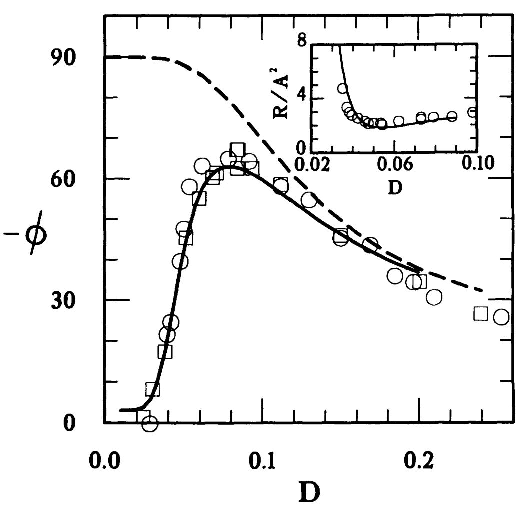

FIG. 1. The phase shift $-\phi$ (degrees) between the periodic force of amplitude A and the averaged coordinate $\langle q(t)\rangle$ measured as a function of noise intensity D in the electronic experiment for $\Omega = 0.1$ and $A = 0.04$ (circles) and $A = 0.2$ (squares). The solid curve is a theoretical prediction based on LRT and the fluctuation dissipation theorem; the dashed curve represents the prediction (13) of earlier (two-state) theories [5–7] that make no explicit allowance for the effect of intrawell vibrations. The inset compares experiment and LRT prediction for the normalized signal-to-noise ratio in the region of the minimum in R.

It is reasonable to wonder why the variation of ϕ with D for SR in model (1) should be *nonmonotonic* (Fig. 1), whereas the corresponding variation of ϕ with Ω in a deterministic resonance is well known to be monotonic (as were also the earlier predictions [5–7] for SR). An answer is readily inferred by physical intuition. For very small $D \ll \Delta U$, where the system is effectively confined to a single well, we may expect ϕ to be small because Ω is small compared with the reciprocal characteristic time of intrawell motion; for very large $D \gg \Delta U$, where the double-well character of the potential has become irrelevant, we may also expect $\phi \simeq 0$, for the same reason; so, at the intermediate values of D where SR occurs, any significant phase lag associated with the SR must inevitably give rise to a maximum in $|\phi|$, just as observed. Earlier theories failed to predict this behavior because, unlike the discussion above, they are effectively two-state treatments that take no account of the intrawell vibrations. This is easily seen because, if we now consider only the interwell transitions, (7) and (10) yield

$$\mathrm{Re}\chi_{\mathrm{tr}}(\Omega) = \frac{1}{D} w_1 w_2 \left[\langle q_1\rangle^{(0)} - \langle q_2\rangle^{(0)}\right]^2 \frac{W^{(0)^2}}{W^{(0)^2} + \Omega^2}\,, \qquad (12)$$

$$\mathrm{Im}\chi_{\mathrm{tr}}(\Omega) = \frac{1}{D} w_1 w_2 \left[\langle q_1\rangle^{(0)} - \langle q_2\rangle^{(0)}\right]^2 \frac{W^{(0)}\Omega}{W^{(0)^2} + \Omega^2}\,,$$

from which, using (4), we immediately obtain the original Nicolis [5] result

$$\phi = -\arctan(\Omega/W^{(0)}) \qquad (13)$$

shown by the dashed curve in Fig. 1.

Although the observation of the phase lags (above and in [11]) certainly strengthens the analogy between SR and conventional forms of resonance, we stress that *it is only an analogy* in the case of (1), because [3] there is no matching of Ω to any internal characteristic frequency of the system. This is to be contrasted with SR in underdamped monostable systems [18], which is a true resonance phenomenon where external noise is used to tune the natural oscillation frequency of the system to that of the periodic force.

The excellent agreement between the LRT prediction and the experimental phase lag measurements in Fig. 1 can be taken as a vindication of our suggestion [2] that LRT provides a useful approach to the SR problem. It has both advantages and disadvantages, as compared to other theoretical treatments. The main disadvantage is that it will only yield quantitatively accurate results when A is small enough for the system to be within its regime of linear response; but this is, of course, a condition that is often fulfilled in practice. It should be noted, however, that the quality of the results obtained will naturally depend on the accuracy of the $Q^{(0)}(\omega)$ used in (7) for the calculation of $\chi(\Omega)$: Use of a poor approximation for $Q^{(0)}(\omega)$ must inevitably result in a correspondingly poor approximation for the predicted response [19].

The advantages of the LRT approach to SR are numerous. In addition to the simplicity of the linear-response formalism, we may note the following: (a) This approach brings a seemingly arcane and very complicated phenomenon within the general context of standard statistical physics; (b) unlike any other theory, LRT provides a quantitative description of R near its minimum at small D, because it is able to treat the intrawell vibrations explicitly; (c) for the same reason, only LRT describes correctly the phase shifts that we have shown (above) to occur in SR; (d) LRT is as easily applied to underdamped systems [3] as it is to overdamped systems such as (1); (e) for systems that are in thermal equilibrium or quasiequilibrium, LRT makes it possible to predict the onset of SR solely on the basis of experimental measurements of $Q^{(0)}(\omega)$ and its evolution with temperature (noise intensity), even in cases for which the response cannot be calculated (e.g., because there is no simple theoretical model of the system under study); (f) the predictive power of the LRT approach is enormously greater than that of earlier theories of SR and has led, for example, to the discovery [18,20] of quite new kinds of SR in diverse classes of systems that probably would not otherwise have been suspected of harboring the phenomenon at all.

In conclusion, we have shown that phase lags indeed occur in SR, but that the positions of the maxima of ϕ and R as functions of D do not coincide. This result immediately resolves a long-standing controversy and adds a further dimension to the analogy (cf. Ref. [2]) between SR and conventional (i.e., deterministic) resonance phenomena. The excellent agreement obtained between experiment and the LRT prediction of the phase lag strongly supports the contention [2] that SR may properly, and usefully, be considered as a linear-response phenomenon within the conceptual framework of standard statistical physics.

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