Action for the Hot Gluon Plasma Based on the Chern-Simons Theory

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We show that the generating functional for hard thermal loops with external gluons in QCD is essentially given by the eikonal for a Chern-Simons gauge theory. This action, determined essentially by gauge-invariance arguments, also gives an efficient way of obtaining the hard thermal loop contributions without the more involved calculation of Feynman diagrams.

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The Chern-Simons (CS) action made its appearance in physics literature over ten years ago as a mass term for gauge fields in three dimensions [1]. Studies since then have revealed many interesting properties of this action. The Abelian version can be used for spin transmutation, converting spin-zero bosons into anyons, for example [2]. The correlators of Wilson lines in a pure CS theory are related to the polynomial invariants of knot theory [3]. Pure CS theory is also closely related to conformal field theory and the Wess-Zumino-Witten (WZW) action in two dimensions [3,4]. Also, actions related to the CS action can be used for self-dual gauge fields and integrable systems [5]. Finally there is an intriguing class of vortex solutions in spontaneously broken CS theory [6]. However, despite this bounty of interesting results there have not been many realistic physical systems for which the CS action is relevant. In this Letter we show that the CS action, more precisely its eikonal, is part of the effective action for describing the gluon plasma in quantum chromodynamics (QCD). This action, determined essentially by gauge-invariance requirements, gives an efficient way of obtaining the hard thermal loop contributions, without having to calculate the corresponding Feynman diagrams. The CS connection is particularly interesting in view of the possibility of producing the quark-gluon plasma in heavy-ion collisions in the near future.

We consider QCD at temperatures well into the deconfinement phase; i.e., we have a "hot" plasma of gluons. The effective action mentioned above is more precisely defined as follows. Pisarski has shown that a partial resummation of Feynman diagrams in thermal QCD is necessary to obtain gauge-invariant results, for example, for the gluon damping rate in the plasma [7]. The resummation amounts to the following. We calculate the one-loop diagrams of thermal QCD; the relevant kinematical regime corresponds to the loop momentum being much larger than the external momenta. These are the so-called hard thermal loop contributions. For these, the external momenta are typically of the order of gT where g is the coupling constant and T is the temperature; where the loop momentum is hard, i.e., at least of the order of T, is the region of interest. The leading contributions are proportional to T^2. The generating functional for hard thermal loops is the effective action. Thus, once the high-temperature contributions of the hard thermal loops have been obtained, calculations can be done for any process starting from the effective action. The result is gauge invariant and consistently accounts for all terms of a given order in coupling constant [7].

Many authors [7,8] have written down versions of this generating functional. The results involve a null vector $Q_\mu=(1,\mathbf{Q})$ and integration $\int d\Omega$ over the directions of the unit vector $\mathbf{Q}$. Diagrammatically, $\mathbf{Q}$ arises as follows. Thermal loops describe the absorption and emission of particles from the surrounding medium or thermal bath. These particles are on mass shell and thus the loop integration is only integration over the momentum three-vector with a distribution of the Bose-Einstein or Fermi-Dirac form. Integration over the magnitude of this vector is then carried out leaving the angles, described by the unit vector $\mathbf{Q}$, for the final integration. We shall also need the following coordinates $(u,v,\mathbf{x}_T)$: $u=\frac{1}{2}Q'\cdot x$, $v=\frac{1}{2}Q\cdot x$, $\mathbf{Q}\cdot\mathbf{x}_T=0$, where $Q'_\mu=(1,-\mathbf{Q})$. The components of the gauge field along Q will be denoted by A^a, viz., $A_\mu=-it^aA^a_\mu$, $Q\cdot A^a=A^a$, $Q\cdot A=A$. $\{t^a\}$ are a basis of the Lie algebra of the group G, chosen as Hermitian matrices in the fundamental representation with $\mathrm{Tr}(t^at^b)=\frac{1}{2}\delta^{ab}$. Following Ref. [8], the generating functional for hard thermal loops with external gluons has the structure

$$\Gamma=\frac{CT^2}{12\pi}\left(2\pi\int d^4x\,A^a_0A^a_0+\int d\Omega\,W\right). \tag{1}$$

$C=C_q$ for quark loop contributions and $C=C_G$ for gluon-ghost loop contributions; C_q and C_G are the quadratic Casimirs for the quark and adjoint representations, respectively. We shall not display the coupling constant g in what follows, as it can be recovered by $A_\mu\to gA_\mu$. The first term in parentheses is the well-known mass term for the time-component of the gauge field. The second term may be considered as what is necessary to render Γ gauge invariant. The information given by the diagrammatic analysis of hard thermal loops is that it can be written as $\int d\Omega\,W$, where W is a functional of $A=Q_\mu A_\mu$.

The condition for the gauge invariance of Γ in (1) is

$$\int d\Omega\,\delta W=4\pi\int d^4x\,\dot{A}^a_0\omega^a\,, \tag{2}$$

where $\delta A_\mu=\partial_\mu\omega+[A_\mu,\omega]$. Equation (2) is realized by

$$\delta W=\int d^4x\,\dot{A}^a\omega^a\,. \tag{3}$$

One can check that (3) is indeed the way gauge invariance is realized, again by analysis of diagrams. Thus it is Eq. (3) that we must solve. We rewrite this, using the transformation law for A^a, as

$$\frac{\partial f}{\partial u}+[A,f]=-\frac{1}{2}\frac{\partial A}{\partial v}, \tag{4}$$

where $f=\delta W/\delta A+\frac{1}{2}A$. Our analysis so far parallels Ref. [8]. However, we shall now solve (3),(4) in terms of the eikonal for the Chern-Simons theory [3,4]. (A solution, which is essentially the same as ours, but in a nonthermal context, is also given in Ref. [9].)

We begin by briefly recalling some aspects of the pure Chern-Simons theory [3,4]. Consider the CS action

$$S=\frac{k}{4\pi}\int_{R^3} d^3x\,\mathrm{Tr}(a_\mu\partial_\nu a_\alpha+\tfrac{2}{3}a_\mu a_\nu a_\alpha)\epsilon^{\mu\nu\alpha}. \tag{5}$$

Here a_μ is the Lie algebra valued gauge potential, $a_\mu=-it^a a_\mu^a$. We shall use complex coordinates $z,\bar z$, $z=x+iy$, to describe the spatial dimensions. The time component a_0 can be set to zero as a gauge choice. The equations of motion then tell us that $a_z, a_{\bar z}$ are independent of time but satisfy the constraint

$$\partial_z a_{\bar z}-\partial_{\bar z}a_z+[a_z,a_{\bar z}]=0. \tag{6}$$

This can be solved for $a_{\bar z}$ as a function of a_z, at least as a power series in a_z. The result is

$$a_{\bar z}=\sum(-1)^{n-1}\int\frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi}\frac{a_z(z_1,\bar z_1)\cdots a_z(z_n,\bar z_n)}{(\bar z-\bar z_1)(\bar z_1-\bar z_2)\cdots(\bar z_n-\bar z)}. \tag{7}$$

This can be checked easily using $\partial_z[1/(\bar z-\bar z')]=\pi\delta^{(2)}(z-z')$.

Define the functional $I[a_z]$ such that

$$\delta I=\frac{ik}{\pi}\int d^2x\,\mathrm{Tr}(a_{\bar z}[a_z]\delta a_z). \tag{8}$$

Since $a_{\bar z}$ is conjugate to a_z, I so defined is the action evaluated for the classical motion; thus it is Hamilton's principal function or the eikonal for the CS action. [Recall that the eikonal for one-dimensional particle mechanics is the integral of $p\,dx$, where p is the canonical momentum, specified as a function of x by fixing the energy. We have an analogous situation where $a_{\bar z}$ is specified in terms of a_z by Eq. (6).] We can write I as

$$I=ik\sum\frac{(-1)^n}{n}\int\frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi}\frac{\mathrm{Tr}[a_z(z_1,\bar z_1)\cdots a_z(z_n,\bar z_n)]}{\bar z_{12}\bar z_{23}\cdots\bar z_{n1}}, \tag{9}$$

where $\bar z_{ij}=\bar z_i-\bar z_j$. I is in fact the WZW action. If we parametrize the two-dimensional gauge field as $a_z=-\partial_z U U^{-1}$, where U is valued in the complexification of the group G, (9) can be written in the more conventional form of the WZW action [10] as $I=-ikS_{\mathrm{WZW}}$, where

$$S_{\mathrm{WZW}}=\frac{1}{2\pi}\int d^2x\,\mathrm{Tr}(\partial_z U\partial_{\bar z}U^{-1})-\frac{i}{12\pi}\int_{M^3}d^3x\,\mathrm{Tr}(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\alpha U)\epsilon^{\mu\nu\alpha}. \tag{10}$$

(As usual $M^3=R^2\times[0,1]$ with $U(z,\bar z,0)=1$, $U(z,\bar z,1)=U(z,\bar z)$.) The WZW action is thus the eikonal for the CS action. For our discussion below, a_μ will not be just a two-dimensional gauge field. Equation (9) will thus be the more useful form. Also, since k is not relevant for our discussion, we shall henceforth set it to 1.

Returning to Eq. (4), notice that it is of the form of a zero-curvature condition. We first do a Wick rotation to Euclidian space so that $2u\to z$, $2v\to\bar z$, $\partial_u\to 2\partial_z$, $\partial_v\to 2\partial_{\bar z}$. Defining $a_{\bar z}=-f$ and $a_z=\frac{1}{2}A$, Eq. (4) is seen to be identical to (6). Hence the solution for W is given by

$$W=-\tfrac{1}{4}\int d^4x\,A^aA^a-4\pi i I[A/2], \tag{11}$$

where from (9) and the definition of a_z,

$$I[A/2]=i\sum\frac{(-1)^n}{n}\int d^2x_T\frac{d^2z_1}{\pi}\cdots\frac{d^2z_n}{\pi}\frac{1}{2^n}\mathrm{Tr}\frac{[A(x_1)\cdots A(x_n)]}{(\bar z_{12}\bar z_{23}\cdots\bar z_{n1})}. \tag{12}$$

The potentials now depend on all four coordinates; however, since (4) does not involve differentiations with respect to the transverse coordinates $\mathbf{x}_T$, all potentials in (12) have the same argument for these coordinates. In other words, the transverse coordinates in (12) only play the role of parameters on which the A's depend. Using (11) in (1) we have the generating functional in terms of the eikonal I,

$$\Gamma=\frac{CT^2}{12\pi}\left[\int d^4x\left(2\pi A_0^aA_0^a-\tfrac{1}{4}\int d\Omega\,A^aA^a\right)-4\pi i\int d\Omega\,I[A/2]\right]. \tag{13}$$

We have not actually calculated Feynman diagrams to arrive at (13). The only input from a diagrammatic analysis has been the structure $\int d\Omega\, W$. However, it is easy to check that the n-point functions calculated from (13) agree with the explicit diagrammatic evaluation of hard thermal loops. The n-point functions in momentum space are given by

$$(2\pi)^4\delta^{(4)}\left(\sum k_i\right)\Gamma^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(k_1\cdots k_n)=\int d^4x_1\cdots d^4x_n \exp\left(-i\sum k_i\cdot x_i\right)V^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n)\,,$$
$$V^{a_1\cdots a_n}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n)=\left[\frac{\delta^n\Gamma}{\delta A^{a_1}_{\mu_1}(x_1)\cdots\delta A^{a_n}_{\mu_n}(x_n)}\right]_{A=0}. \tag{14}$$

The two-point function is given by

$$\Gamma^{ab}_{\mu\nu}(x_1,x_2)=\delta^{ab}\frac{CT^2}{12\pi}\left[4\pi\delta_{\mu0}\delta_{\nu0}\delta^{(4)}(x_1-x_2)-\tfrac{1}{2}\int d\Omega\, Q_\mu Q_\nu\left(\delta^{(4)}(x_1-x_2)-\frac{\delta^{(2)}(x_{T_1}-x_{T_2})}{\pi(\bar z_1-\bar z_2)^2}\right)\right]. \tag{15}$$

We need the Fourier transform to obtain the expression in momentum space. This is straightforward for the first two terms. For the last term we have an expression of the form

$$H=\int\frac{d^2z_2}{\pi}\frac{h(z_2,\bar z_2)}{(\bar z_1-\bar z_2)^2}=-\partial_{\bar z_1}\int\frac{d^2z_2}{\pi}\frac{h(z_2,\bar z_2)}{(\bar z_1-\bar z_2)}\,, \tag{16}$$

where h is an exponential of the form $\exp[i(k_z\bar z+k_{\bar z}z)]$. Using $\partial_z[1/(\bar z-\bar z')]=\pi\delta^{(2)}(z-z')$, we get

$$\partial_{z_1}H=-\partial_{\bar z_1}h\,. \tag{17}$$

This can be easily solved for H. The Fourier transform of (15), after Wick rotation to Minkowski space, with $2k_{\bar z}\to k\cdot Q$, $2k_z\to k\cdot Q'$, gives

$$\Gamma^{ab}_{\mu\nu}=\delta^{ab}\frac{CT^2}{12\pi}(4\pi\delta_{\mu0}\delta_{\nu0}-f_{\mu\nu})\,, \tag{18}$$

where $f_{\mu\nu}=\int d\Omega\, Q_\mu Q_\nu k_0/k\cdot Q$.

The three-point function involves the factor $(\bar z_{12}\bar z_{23}\bar z_{31})^{-1}$ in addition to the transverse δ function and color and Q_μ factors. Using the splitting

$$\frac{1}{\bar z_{12}\bar z_{23}\bar z_{31}}=\frac{1}{(\bar z_{12})^2}\left(\frac{1}{\bar z_{13}}-\frac{1}{\bar z_{23}}\right) \tag{19}$$

the Fourier transform can be evaluated by the same method as in (16) and (17) to obtain

$$\Gamma^{abc}_{\mu\nu\lambda}=f^{abc}\frac{iCT^2}{12\pi}\int d\Omega\, Q_\mu Q_\nu Q_\lambda\frac{1}{k_3\cdot Q}\left(\frac{k_{20}}{k_2\cdot Q}-\frac{k_{10}}{k_1\cdot Q}\right). \tag{20}$$

Expressions (18) and (20) agree with the diagrammatic evaluation of hard thermal loops [7,11]. We have checked the four-point function in a similar way. For this and for the higher point functions, a splitting formula analogous to (19) is very useful. It is given by the Ward identity or in other words, by the recursive buildup of the correlator of currents in the WZW model. Combined with the Fourier transform method we have used, this gives an efficient way of calculating the higher point functions.

The vectors Q,Q', in terms of the coordinates u,v, define a two-dimensional subspace in spacetime. Our results indicate that at high temperature, as far as the hard thermal loops are concerned, the dynamics is essentially the CS dynamics on this subspace, i.e., for the components A_u,A_v. The choice of this subspace can be incorporated into the action by using, instead of (5),

$$S=\frac{k}{4\pi}\int_{R^5}d^5x\,\mathrm{Tr}(a_\mu\partial_\nu a_\alpha+\tfrac{2}{3}a_\mu a_\nu a_\alpha)\omega_{\beta\tau}\epsilon^{\mu\nu\alpha\beta\tau}\,, \tag{21}$$

where $\omega_{\mu\nu}=\frac{1}{2}(m_\mu n_\nu-m_\nu n_\mu)$, with m_μ,n_ν defining a basis for vectors transverse to the Q-Q' plane, with $m_5,n_5=0$. This action is similar to the Kähler-Chern-Simons (KCS) action considered in Ref. 5. The difference is that for us ω, being restricted to directions transverse to the Q-Q' plane, is degenerate. As for the KCS theory, the equations of motion tell us that the fields do not depend on the extra fifth dimension in the action (21). Our final results, of course, do not depend on the choice of the subspace defined by $\mathbf{Q}$ since we integrate over the orientations of $\mathbf{Q}$.

Since the CS action is odd under parity, its presence in a QCD calculation may be potentially worrisome. However, we do not have any parity violation because all our results are integrated over the orientations of $\mathbf{Q}$. Only the parity-preserving contributions to the n-point functions survive this integration. Also for the non-Abelian CS action, one has to address the issue of quantization of the coefficient of the action. Again, the integration over the orientations of $\mathbf{Q}$ shows that there is no quantization. The quantization arises, in the usual analysis, from the requirement of invariance under homotopically nontrivial gauge transformations. In our case, there are no nontrivial gauge transformations consistent with the angular symmetry imposed by the integration over the orientations of $\mathbf{Q}$. In other words, the relevant winding number corresponding to maps of the three-sphere into the group cannot be defined in a way that is invariant under the $\mathbf{Q}$ integration.

We conclude by rewriting the action in another way. We define $a_{\bar z}=\frac{1}{2}Q'\cdot A$, $a_z=\frac{1}{2}Q\cdot A$; $a_{\bar z}$ and a_z are independent and no longer related by Eq. (6). Further, define $\mathbf{q}=q\mathbf{Q}$; we can then write

$$\Gamma=8\pi\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q}N(q)K[a_z,a_{\bar{z}}]\,,$$
$$K[a_z,a_{\bar{z}}]=-2C\left[\frac{1}{\pi}\int d^4x\,\mathrm{Tr}(a_z a_{\bar{z}})+iI+i\tilde{I}\right]. \tag{22}$$

Here $N(q)$ is the Bose-Einstein distribution (for gluon loops) and $\tilde{I}$ is given by I [from (12)] with the change $a_z \to a_{\bar{z}}$, $z \to \bar{z}$. The density K is gauge invariant; apart from integration over the transverse coordinates, it can be written as $\mathrm{Tr}\ln(D_z D_{\bar{z}})$, where $D_z, D_{\bar{z}}$ are the corresponding two-dimensional covariant derivatives in the adjoint representation [9,12]. [For quark loop contributions, there is a similar formula with the Fermi-Dirac distribution for $N(q)$.] K can also be considered as the Kähler potential associated to the symplectic structure for the CS action [3,4]. Following Ref. [13], it may also be possible to relate this to the forward scattering amplitude for high-energy gluons on a gauge field background. These issues will be discussed in more detail elsewhere.

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