Coupled Nonlinear Oscillators below the Synchronization Threshold: Relaxation be Generalized Landau Damping

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We analyze a model of globally coupled nonlinear oscillators with randomly distributed frequencies. Twenty-five years ago it was conjectured that, for coupling strengths below a certain threshold, this system would always relax to an incoherent state. We prove this conjecture for the system linearized about the incoherent state, for frequency distributions with compact support. The relaxation is exponentially fast at intermediate times but slower than exponential at long times. The decay mechanism is remarkably similar to Landau damping in plasmas, even though the model was originally inspired by biological rhythms.

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Nonlinear oscillators are among the oldest and best understood types of dynamical systems, yet little is known about their collective behavior. Recently there has been a great deal of interest in coupled oscillators, in part because they arise in many branches of science, and also because of a broader interest in high-dimensional dynamical systems [1-9].

The problem studied in this Letter was originally inspired by the biological phenomenon of mutual synchronization [1]. In some parts of southeast Asia, thousands of male fireflies gather in trees at night and flash on and off in unison. Other examples include chorusing of crickets, synchronous firing of cardiac pacemaker cells, and metabolic synchrony in yeast cell suspensions [1]. A simple model of such systems consists of a population of coupled phase-only oscillators with distributed natural frequencies. The governing equations [2] are

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad (1)$$

for $j=1,\ldots,N \gg 1$. Here θ_i is the phase of oscillator i, ω_i is its natural frequency, and $K \geq 0$ is the coupling strength. The frequencies are randomly chosen from a probability density $g(\omega)$, assumed to be one-humped and symmetric about its mean. By choosing a rotating frame at the mean frequency, we may assume that $g(\omega)$ has mean zero. For simplicity, the coupling in (1) is all-to-all, corresponding to mean-field theory as $N \to \infty$.

Early studies [1,2] of Eq. (1) revealed a beautiful connection to equilibrium statistical mechanics: A phase transition occurs at a critical coupling given by $K_c = 2/\pi g(0)$. For $K < K_c$, the system relaxes to an incoherent state with each oscillator running at its natural frequency, but for $K > K_c$, mutual synchronization occurs spontaneously in a small group of oscillators. This transition can be described by a complex order parameter $re^{i\psi} = N^{-1}\sum_j e^{i\theta_j}$, where r measures the phase coherence of the population. In the limit of infinite N, Kuramoto [2] determined the steady values of r self-consistently. He showed that incoherence ($r=0$) is always a steady solution, but a branch of partially locked solutions ($r>0$) bifurcates supercritically from incoherence at K_c.

It has turned out to be a much more delicate matter to analyze the *stability* of the steady states. In particular, there has been a controversy about the relaxation to incoherence for $K < K_c$. Kuramoto and Nishikawa [8] have proposed two theories, one predicting algebraic relaxation of the order parameter $r(t)$, and the other predicting exponential relaxation. Numerical simulations show an approximately exponential relaxation, at least for K close to K_c. Yet this result seems paradoxical in view of our recent proof [9] that the incoherent state is linearly *neutrally* stable below threshold, and that there are no exponentially decaying eigenfunctions in this case.

In this Letter we clarify the subthreshold behavior of Eq. (1). We show that the order parameter r can decay, even though the incoherent state is neutrally stable. The decay mechanism is closely related to Landau damping of waves in collisionless plasmas [10], a phenomenon which created its own share of confusion about thirty years ago. Our analysis accounts for the exponential decay observed near K_c, but shows that for $g(\omega)$ with compact support, this decay is confined to intermediate times; at long time, the decay is slower than exponential.

We begin by writing the dynamics in the infinite-N limit. Intuitively, one should imagine each oscillator as a particle moving around a circle. For each frequency ω, let $\rho(\theta,t,\omega)$ denote the density of oscillators at angle θ at time t, and let $v(\theta,t,\omega)$ denote the local velocity. Then ρ satisfies the continuity equation $\partial\rho/\partial t = -\partial(\rho v)/\partial\theta$; this merely expresses conservation of oscillators of frequency ω. Here the velocity $v(\theta,t,\omega)$ is given by

$$v = \omega + K\int_0^{2\pi}\int_{-\infty}^{\infty} \sin(\phi-\theta)\rho(\phi,t,\omega)g(\omega)\,d\omega\,d\phi\,,$$

where we have used the law of large numbers to replace the sum in Eq. (1) by an integral. [Similarly, the order parameter becomes $re^{i\psi} = \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\rho(\theta,t,\omega)g(\omega)\,d\omega$

$d\theta$.] Thus we have a nonlinear integro-partial-differential equation for ρ. We also require ρ to be 2π periodic in θ, with $\rho \geq 0$ and $\int_0^{2\pi} \rho \, d\theta = 1$.

Ultimately one would like to understand the global dynamics of this system. For now we restrict ourselves to a linear analysis about the fixed point $\rho(\theta,t,\omega) \equiv 1/2\pi$, which corresponds to the "incoherent state." Let

$$\rho = 1/2\pi + \varepsilon[c(t,\omega)e^{i\theta} + (\text{complex conjugate}) + (\text{higher harmonics})],$$

where $\varepsilon \ll 1$. We write the perturbation in this way because the order parameter depends only on c, and not on the higher harmonics:

$$r(t) = 2\pi\varepsilon \left| \int_{-\infty}^{\infty} c(t,\omega) g(\omega) d\omega \right|. \tag{2}$$

Furthermore, at $O(\varepsilon)$ the amplitude equation for $c(t,\omega)$ decouples from the other harmonics:

$$\frac{\partial c}{\partial t} = -i\omega c + \frac{K}{2} \int_{-\infty}^{\infty} c(t,\nu) g(\nu) d\nu. \tag{3}$$

Hence we may ignore the higher harmonics altogether.

The right-hand side of Eq. (3) defines a linear operator A which has both a discrete and a continuous spectrum. The discrete spectrum is given by solutions to $(K/2)\int_{-\infty}^{\infty} (\lambda + i\omega)^{-1} g(\omega) d\omega = 1$. For $g(\omega)$ even and nonincreasing on $[0,\infty)$, there is either no solution for λ (for $K \leq K_c$), or a unique, positive real solution which tends to zero as $K \to K_c^+$. In particular, A never has negative eigenvalues. The continuous spectrum exists for all K, is pure imaginary, and is given by $\{i\omega : g(\omega) \neq 0\}$. These facts [9] show that the incoherent state is linearly neutrally stable below threshold.

Our new results concern the subthreshold behavior of $r(t)$. Since the integral that appears in Eq. (3) is so closely related to $r(t)$ by Eq. (2), we introduce the notation

$$R(t) = \int_{-\infty}^{\infty} c(t,\omega) g(\omega) d\omega. \tag{4}$$

Note that Eq. (3) may be solved easily for $c(t,\omega)$ in terms of $R(t)$ and the initial condition $c_0(\omega) \equiv c(0,\omega)$. When the result is combined with (4), we obtain the linear integral equation

$$R(t) = (\widehat{c_0 g})(t) + \frac{K}{2} \int_0^t R(t-\tau) \hat{g}(\tau) d\tau, \tag{5}$$

where the hat denotes the Fourier transform: $\hat{g}(t) = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega$. Hence our problem reduces to understanding $R(t)$.

The asymptotic behavior of $R(t)$ depends crucially on whether $g(\omega)$ is supported on a finite interval $[-\gamma,\gamma]$, or the whole real line (these are the only possibilities, by our hypotheses on g). We focus on the simpler case of $g(\omega)$ with compact support; the case of infinite support will be discussed briefly near the end of this paper.

Assume from now on that $K < K_c$ and that $g(\omega)$ has compact support $[-\gamma,\gamma]$. Then A is a *bounded* linear operator; hence $c(t,\omega)$ and $R(t)$ are both analytic in t. Moreover, these functions grow no faster than exponentially in t, and so the Laplace transforms $\tilde{c}(s,\omega)$ and $\tilde{R}(s)$ are well defined for $\mathrm{Re}(s)$ sufficiently large. Now we Laplace transform (3) and (4) and solve for $\tilde{R}(s)$. The result is

$$\tilde{R}(s) = \frac{(c_0 g)^*(s)}{1 - (K/2) g^*(s)}, \tag{6}$$

where the asterisk denotes an operation related to the Hilbert transform: $f^*(s) \equiv \int_{-\infty}^{\infty} f(\omega) d\omega / (s + i\omega)$.

The long-term behavior of $R(t)$ is controlled by the singularities of $\tilde{R}(s)$. For $g(\omega)$ supported on a finite interval, these singularities will include branch points as well as poles. Equation (6) shows that $\tilde{R}(s)$ is analytic and single valued in the region $\mathbf{C} - i[-\gamma,\gamma]$, i.e., away from the continuous spectrum of A. To see this, note first that the denominator of $\tilde{R}(s)$ vanishes precisely when s is in the discrete spectrum of A. Hence, for $K < K_c$, the denominator *never* vanishes. Second, the functions $(c_0 g)^*$ and g^* are analytic in the region $\mathbf{C} - i[-\gamma,\gamma]$, because

$$(c_0 g)^*(s) = \int_{-\gamma}^{\gamma} (s + i\omega)^{-1} c_0(\omega) g(\omega) d\omega$$

defines $(c_0 g)^*$ unambiguously as a single-valued analytic function in this region.

Now we can obtain our first theorem: Suppose that $g(\omega)$ has compact support $[-\gamma,\gamma]$. Then for any nonzero initial perturbation $c_0(\omega)$, $R(t)$ decays more slowly than any exponential as $t \to \infty$. The proof is by contradiction, and involves analytic continuation and Liouville's theorem. Suppose that $|R(t)| \leq Ce^{-at}$ for all $t > 0$, for some $a > 0$. Then $\tilde{R}(s)$ would be analytic in the region $\mathrm{Re}(s) > -a$, by a standard theorem about Laplace transforms. Hence the left-hand side of Eq. (6) would be analytic in the region $\mathrm{Re}(s) > -a$, whereas the right-hand side is analytic in the region $\mathbf{C} - i[-\gamma,\gamma]$. These two regions overlap on an open set, and their union is the whole complex plane $\mathbf{C}$; by analytic continuation, $\tilde{R}(s)$ must be analytic on *all* of $\mathbf{C}$, i.e., entire. But Eq. (6) shows that $\tilde{R}(s) \to 0$ as $|s| \to \infty$ [since $(c_0 g)^*(s) \to 0$ and $g^*(s) \to 0$]. Hence $\tilde{R}(s)$ is a bounded, entire function which vanishes as $|s| \to \infty$. By Liouville's theorem, $\tilde{R}(s) \equiv 0$. This implies that $R(t) = 0$ for all t, since R is analytic in t. Then (5) implies $(\widehat{c_0 g})(t) \equiv 0$. Hence $c_0 g = 0$ and so $c_0 = 0$ for all ω in the support of g. But this contradicts our assumption of a nonzero initial perturbation. Hence $R(t)$ cannot be bounded above by any decaying exponential.

Nevertheless, $R(t)$ does tend to zero as $t \to \infty$. The proof is technical, so we outline the ideas. First one uses the Paley-Wiener theorem [11] to show that if $c_0 g \in L^2$ and $K < K_c$, then $R \in L^2$. Hence $R = \hat{h}$, for some $h \in L^2$. Now the key step is to show that this h has *finite support* lying inside $[-\gamma,\gamma]$, and so $R(t)$

$=\int_{-\gamma}^{\gamma} h(\omega) e^{-i\omega t} d\omega$. Then the Riemann-Lebesgue lemma yields $R(t) \to 0$, as desired. The form of this integral representation for $R(t)$ should seem plausible—it essentially follows from the Laplace inversion formula applied to Eq. (6). One chooses a finite branch cut on the imaginary axis between $\pm i\gamma$, and then wraps the inversion contour around this cut.

By manipulating the inversion contour, we can also explain the exponential decay observed numerically for K just below K_c. At first such behavior is puzzling, given the theorem above and the absence of exponentially decaying eigenfunctions. The explanation involves "fake eigenvalues" that arise via analytic continuation [12]. For concreteness, let $c_0(\omega) \equiv 1$. Then

$$R(t) = \frac{1}{2\pi i} \int_\Gamma \frac{g^*(s)}{1-(K/2)g^*(s)} e^{st} ds , \qquad (7)$$

where the contour Γ is a vertical line to the right of the imaginary axis. Now choose the branch cuts of g^* as shown in Fig. 1, and deform the contour to Γ', as shown. In order for this deformation to be valid, we need $\tilde{R}(s)$ to be analytic in the region between Γ and Γ'. Hence Eq. (7) requires us to consider not g^*, but its *analytic continuation* $G(s)$, given by $G(s) = g^*(s)$, for $\mathrm{Re}(s) > 0$; $G(s) = g^*(s) + \pi g(is)$ for $\mathrm{Re}(s) = 0$; and $G(s) = g^*(s) + 2\pi g(is)$, for $\mathrm{Re}(s) < 0$. For $K < K_c$, the resulting analytic continuation of $\tilde{R}(s)$ has poles in the left half plane given by the solutions of $1-(K/2)G(s)=0$; these poles are the fake eigenvalues that contribute exponentially decaying terms to $R(t)$.

For instance, consider the uniform density $g(\omega) = 1/2\gamma$ on $[-\gamma,\gamma]$. Then $G(s) = \gamma^{-1}\arctan(\gamma/s) + \pi/\gamma$ for $\mathrm{Re}(s) < 0$. Hence for $K < K_c = 4\gamma/\pi$, there is a single fake eigenvalue at $s = \gamma\cot(2\gamma/K) < 0$. This predicted exponential decay rate agrees well with that observed numerically [Fig. 2(a)] [13]. $R(t)$ also contains contributions obtained from integration along the branch cuts; these yield slowly damped oscillatory terms with frequency γ, as also seen numerically [Figs. 2(a) and 2(b)]. At long times, the decay is slower than exponential (as expected) and is well fitted by a power law $t^{-\beta}$ with $\beta > 1$ [Fig. 2(b)]. Asymptotic analysis of (7) for uniform $g(\omega)$ shows that $R(t) \sim -16\gamma K^{-2}(t\ln^2 t)^{-1}\sin\gamma t$ as $t \to \infty$ [14]. This formula correctly predicts damped oscillations at frequency γ, with a decay slightly faster than $1/t$, but it is not quantitatively accurate until t is very large.

Different behavior occurs if $g(\omega)$ is supported on the whole real line. In particular, pure exponential decay becomes possible. For instance, if $c_0(\omega) \equiv 1$ and $\hat{g}(t) = e^{-|\gamma t|}$, corresponding to a Lorentzian $g(\omega) = (\gamma/\pi)(\gamma^2 + \omega^2)^{-1}$, then Eq. (5) has the exact solution $R(t) = e^{(K/2-\gamma)t}$ for $t \geq 0$. In the extremely well-behaved case where $g(\omega)$ and $c_0(\omega)$ are entire functions, the contour deformation argument above shows that $R(t)$ is a sum of decaying exponentials. More complicated behavior can arise if the initial conditions are less smooth; e.g., if we demand only that $c_0 \in L^2$, then *any* $R(t) \in L^2$ can be contrived by an appropriate choice of c_0 [14].

The behavior discussed here is closely analogous to Landau damping of waves in collisionless plasmas [10]. The distribution over natural frequencies in our oscillator model corresponds to a distribution over velocities in the plasma model. The mean-field nature of the plasma problem is due to the fact that the individual particles respond to the electric field which is generated by an integral over all the particles. Hence the electric field plays the role of our order parameter. To see the analogy in more detail, we briefly recall the Vlasov model for a plasma. For particles of mass m and charge e, the collisionless Boltzmann equation in one dimension is

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{e}{m}E\frac{\partial F}{\partial v} = 0 ,$$

where $F(x,v,t)$ is the number density of particles with velocity v at position x, and E is the electric field. Suppose we have an equilibrium in which $E=0$ and $F(x,v,t) = F_0(v)$. Then introducing a density perturbation f and linearizing yields

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}E\frac{\partial F_0}{\partial v} = 0 ,$$

where E is determined self-consistently by Poisson's equation

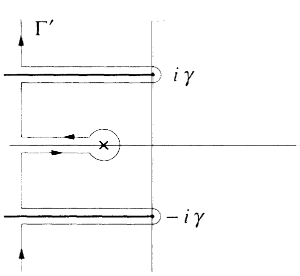

FIG. 1. Deformed contour Γ' for Laplace inversion integral (7), for $g(\omega)$ with compact support $[-\gamma,\gamma]$. Heavy lines, branch cuts; cross, pole.

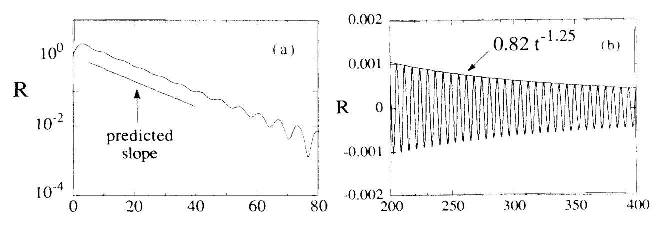

FIG. 2. Numerical solution for $R(t)$ in Eq. (5), assuming uniform $g(\omega)$, $\gamma=1$, $K=0.95K_c \approx 1.21$, and $c_0(\omega) \equiv 1$. (a) $R(t)$ decays exponentially for $10 \leq t \leq 80$. For comparison, line segment shows predicted decay rate $s = \gamma\cot(2\gamma/K) = -0.0829$. The oscillations in $R(t)$ have frequency $=\gamma$. (b) At long times, $R(t)$ decays slower than exponentially.

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0} \int_{-\infty}^{\infty} f \, dv \, .$$

Fourier transforming these equations in space and eliminating $\hat{E}$ yields

$$\frac{\partial \hat{f}}{\partial t} = -ikv\hat{f} + \frac{ie^2}{km\varepsilon_0} \frac{\partial F_0}{\partial v} \int_{-\infty}^{\infty} \hat{f} \, dv \, . \tag{8}$$

Equation (8) is very similar to Eq. (3), with $\hat{f}$ and v playing the roles of c and ω, respectively. The analysis of Eq. (8) reveals many of the same peculiar features seen here: a continuous spectrum on the imaginary axis, the need for analytic continuation, the existence of fake eigenvalues, and the fact that the electric field can decay exponentially (Landau damping), even though the density perturbation does not [10,12].

The physical interpretation of Landau damping is that one can find a distribution of particles in velocity space which does not decay but which combines as time progresses in such a way that the electric field decays exponentially. Similarly for the oscillator problem, the oscillators can spread out around the circle, becoming more well mixed by the difference in their frequencies, so that the order parameter decays to zero. Thus in both cases the damping is due to phase mixing, modified by a self-consistent decaying field.

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