Distribution of Roots of Random Polynomials

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We consider polynomials of high degree with random coefficients which appear in the context of "quantum chaotic" dynamics and investigate various conditions under which their roots tend to concentrate near the unit circle in the complex plane. Correlation functions of roots are computed analytically. We also investigate a certain class of random polynomials whose roots cover, in a uniform way, the Riemann sphere. Special emphasis is devoted to the influence of symmetries.

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(1) The main purpose of this Letter is to discuss properties of roots of polynomials of high degree with random coefficients. This kind of problem naturally arises in different branches of physics (see, e.g., Ref. [1], and references therein). Our investigation was motivated by the problem of obtaining the eigenvalues and eigenvectors of "quantum chaotic systems."

From a statistical point of view, one usually assumes that the highly excited energy levels of a chaotic quantum system should be distributed as eigenvalues of a certain ensemble of random matrices [2,3]. In proving this conjecture the main difficulty comes from the fact that it does not seem easy to naturally attribute a random matrix to a quantum system. On the other hand, many efforts have been devoted to obtaining the quantum spectrum from semiclassical sums over periodic orbits (see, e.g., Ref. [4]). In one of these approaches, developed in Ref. [5], the dynamical zeta function—whose zeros give the eigenvalues—is written in the form

$$P(z)=\det(1-zT),$$

where T is a $N\times N$ matrix corresponding to the semiclassical reduction of the full quantum problem into a discrete mapping (quantum Poincaré map), and as such it should be a unitary matrix [6]. Expanding the determinant, the characteristic polynomial $P(z)$ takes the form

$$P(z)=\sum_{k=0}^{N}a_k z^k . \tag{1}$$

In the leading $\hbar$ approximation T is approximately unitary; hence the roots of $P(z)$ lie close to the unit circle —denoted by $\mathcal{C}$—in the complex z plane. The coefficients $\{a_k\}$ are related to certain sums over classical periodic orbits [5]. These sums are rapidly varying functions of energy and a simple approximation is to consider them as random variables. We are thus led, through the semiclassical approximation, to consider random polynomials instead of random matrices. The main question is to determine what *simple* conditions have to be imposed on the distribution of the $\{a_k\}$ in order that all or at least a finite fraction of the roots of the polynomial (1) lie on (or near) $\mathcal{C}$ (i.e., the T matrix will be approximately unitary).

A necessary condition is the self-inversive (SI) property

$$a_{N-k}=e^{i\phi}\bar{a}_k , \tag{2}$$

which holds in the semiclassical approximation [5] and in actual computations reduces the number of periodic orbits that must be taken into account. This is by no means a sufficient condition. The exact conditions that all roots of polynomial (1) lie on $\mathcal{C}$ are certain complicated determinantal inequalities for the coefficients [7], which seem to be intractable. In this context, let us mention the theorem of Yang and Lee [8], which is an example of a different mechanism of putting all roots of a polynomial on $\mathcal{C}$.

In part (3) we shall demonstrate that in the limit $N\to\infty$ the vanishing of the first moments of the $\{a_k\}$ and certain conditions of boundedness of their second moments are enough to ensure that most roots lie *near* the unit circle. In part (4) we show that if one imposes the SI condition (2) then a finite fraction of the roots lie *exactly* on the unit circle. Assuming that the real and imaginary parts of the complex coefficients $\{a_k\}$, with $k=1,\ldots,[N/2]$, are independent real-valued Gaussian random variables (henceforth denoted as a GRI distribution), we compute analytically the two-point correlation function of the roots lying on $\mathcal{C}$ and find a linear repulsion among them. Part (5) deals with, in some sense, an opposite problem taken from the investigation of quantum chaotic dynamics of models with a compact phase space [9,10]. As explained below, in an appropriate representation the quantum eigenstates can also be written in the form of Eq. (1), the complex variable z spanning the phase space of the system. Looking for a quantum analog of the classical ergodicity, we now want to find the conditions under which the roots of polynomial (1) spread over the complex plane. It is shown that if the complex coefficients $\{a_k\}$ have a GRI distribution with zero mean and second moments given by $\sigma_k^2=\sigma^2 C_N^k$ (where C_N^k are the binomial coefficients and σ is arbitrary), then the density of roots is uniform on the Riemann sphere for all N. In part (6) we consider a particular model—the quantum top—and show how the existence of a symmetry leads to a concentration of the roots of the eigenfunctions on certain calculable phase-space lines.

We shall discuss each section briefly, and details will be published elsewhere [11].

(2) The main technical tool which we shall repeatedly use is a generalization of Kac's method [1,12] which was originally used to compute the mean number of real roots of a polynomial with random real independent coefficients [12].

Let $f(x)$ be a real-valued random function; we are interested in computing the density of roots,

$$\rho(x) \equiv \sum_k \delta(x-x_k) = \delta(f(x))|f'(x)| .$$

Representing $\delta(x)$ and $|x|$ as

$$\delta(x) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x}, \quad |x| = \int_{-\infty}^{\infty} \frac{d\eta}{\pi\eta^2}(1-e^{i\eta x}),$$

one obtains a very suitable method for computing the mean value of the density of roots because averages over the independent random coefficients of $f(x)$ can be easily performed. Knowing $\rho(x)$, the correlation functions of roots can be calculated from the definition $R_n = \langle \rho(x_1) \cdots \rho(x_n) \rangle$, where the symbol $\langle\ \rangle$ denotes ensemble average over random coefficients. It is not difficult to generalize these expressions to complex-valued functions.

(3) Let us assume that the complex coefficients of polynomial (1) have a distribution function

$$\int D(a_0,a_1,\ldots,a_N) d^2a_0 d^2a_1 \cdots d^2a_N . \tag{3}$$

Transforming variables from the coefficients $\{a_k\}$ to the roots $\{z_k\}$ by standard formulas, one obtains the distribution function of the $\{z_k\}$:

$$\int \mathcal{D}(a_0;z_1,\ldots,z_N)|a_0|^{2N} \times \prod_{j<k} |z_j - z_k|^2 d^2a_0 d^2z_1 \cdots d^2z_N ,$$

where the function $\mathcal{D}(a_0;z_1,\ldots,z_N)$ results from the substitution $\{a_k(z_k)\}$ into the function $D(a_0,a_1,\ldots,a_N)$ and the factor $\prod |z_j - z_k|^2$ comes from the Jacobian of the transformation. This factor plays a crucial role in the correlations of roots of random polynomials.

Suppose now that the distribution of the coefficients (3) has its maximum when all $a_k = 0$. For example, that would be the case if the coefficients were uncorrelated random variables and the distribution function of each coefficient had its maximum at $a=0$. It is possible to show that in the limit $N \to \infty$ the integral (3) has a saddle point when all the coefficients except the first and the last are zero. This corresponds to equally spaced roots lying near $\mathcal{C}$,

$$(z_k)_{\mathrm{sp}} = r_N \exp\left[i \left(\frac{2\pi}{N} k + \phi \right) \right], \quad k = 0,\ldots,N-1 , \tag{4}$$

where $r_N \to 1$ as $N \to \infty$. We shall call this configuration the *crystal*, and the phenomenon of the dominance of this saddle-point configuration, the crystallization of roots on $\mathcal{C}$. This saddle-point solution exists under very general conditions [13], but its influence on the distribution of the roots will mainly depend on the second moments of $\{a_k\}$. If they are small [i.e., the maximum of $D(a_0, a_1,\ldots,a_N)$ is sharp] then the crystallization will be strong, resulting in the attraction of the N roots near $\mathcal{C}$ [1]; if, on the contrary, the second moments are large, the saddle point (4) will be irrelevant and roots may spread over the complex plane (we shall refer to this configuration as the liquid phase). This balance between the crystalline and liquid phases is a general phenomenon when studying roots of random polynomials of high degree.

(4) Thus, the large-N limit permits, under rather general conditions, a concentration of the roots of random polynomials near $\mathcal{C}$ but not, in general, on it. We now explore the consequences on the distribution of roots of the SI symmetry (2), when only half of the coefficients of the polynomial can be chosen freely. Polynomials obeying Eq. (2) satisfy the functional equation

$$\overline{P(1/\bar{z})} = \exp(-i\phi) P(z)/z^N ,$$

from which it follows that if z_p is a root, then $1/\bar{z}_p$ is also a root, i.e., the roots either lie on $\mathcal{C}$ or are symmetrically located under inversion with respect to it. By substitution $z = \exp(i\theta)$, self-inversive polynomials transform into real trigonometric polynomials and Kac's method can be used to compute statistical properties of roots lying on $\mathcal{C}$.

As an example, we consider the SI polynomials of the form

$$P(z) = 1 + \sum_{k=1}^{N-1} a_k z^k + z^N , \tag{5}$$

where the $\{a_k\}$, $k = 1,\ldots,[N/2]$, are GRI-distributed complex variables, all having the same second moment σ^2 and $a_{N-k} = \bar{a}_k$. We compute the average number n of roots of this polynomial lying on $\mathcal{C}$ as a function of σ and find that, to leading order in $1/N$, it only depends on the scaled parameter $\epsilon = \sqrt{N}\sigma$. The fraction $\nu(\epsilon) = n(\epsilon)/N$ of roots lying on $\mathcal{C}$ is given by

$$\nu(\epsilon) = \frac{1}{\sqrt{2\pi}} \int_{-(2/\epsilon^2)^{1/2}}^{(2/\epsilon^2)^{1/2}} dy\, e^{-y^2/2} + \frac{1}{\pi\sqrt{3}} \int_0^1 dx \int_0^\pi d\phi \exp\left[-\frac{1}{\epsilon^2} \left(\cos^2\phi + \frac{3\sin^2\phi}{x^2} \right) \right] .$$

As $\epsilon \to 0$, $\nu(\epsilon) \to 1$ as expected from Eq. (5) since $P(z) \to 1 + z^N$, and the roots are distributed on $\mathcal{C}$ according to the crystal lattice (4). On the other extreme, when $\epsilon \to \infty$, we get $\nu(\epsilon) \to 1/\sqrt{3}$. The "unitarity" implemented by the SI symmetry is thus quite strong, since even in the latter limit the average number of roots remaining on $\mathcal{C}$ does not tend to zero but approaches 58% of the total number. This "macroscopic" number of roots lying on $\mathcal{C}$ because of the SI symmetry should be compared to the $(\ln N)/N$ fraction of real roots obtained by Kac in the case of algebraic polynomials with real random coefficients [12].

We have also computed, for arbitrary σ and N, the two-point correlation function $R_2(\tau)=\langle\rho(\theta)\rho(\theta+\tau)\rangle$ for the set of roots remaining on $\mathcal{C}$. In Fig. 1, R_2 is displayed as a function of the scaled parameter $\delta=\tau N/2\pi$ in the limit $\sigma\to\infty$. For short distances we observe a repulsion between the roots. Notice, however, that $R_2(\tau)$ is different from the random-matrix result for the orthogonal ensemble. In particular, for large N the slope at the origin of R_2 is $\pi^2/10\sqrt{3}$, in contrast to the $\pi^2/6$ result for eigenvalues of random matrices. The long-range behavior shows pronounced oscillations which are remnants of the crystalline structure existing for $\sigma=0$.

(5) Consider now a slightly different class of polynomials, the ones taking the form

$$P(z)=\sum_{k=0}^{N}(C_N^k)^{1/2}a_k z^k\,. \qquad (6)$$

This kind of polynomial arises when considering the quantum mechanics of a spin **S** system whose modulus S is conserved. Using the spin coherent-state representation, it can be shown [14] that an *arbitrary* quantum state of the $(2S+1)$-dimensional Hilbert space can be written as in Eq. (6), where the coefficients $\{a_k\}$ define the state and $N=2S$. The appearance of the factors $(C_N^k)^{1/2}$ in Eq. (6) has a purely kinematical origin and is intimately related to the geometry of phase space, the two-dimensional unit sphere [the radius of the sphere is given by $\hbar\sqrt{S(S+1)}$, which we normalize to unity]. The complex variable z is connected to the variables (θ,ϕ) spanning the Riemann sphere through the stereographic projection $z=\cot(\theta/2)\exp(i\phi)$. Lebœuf and Voros have recently studied [9] the distribution of roots of eigenstates

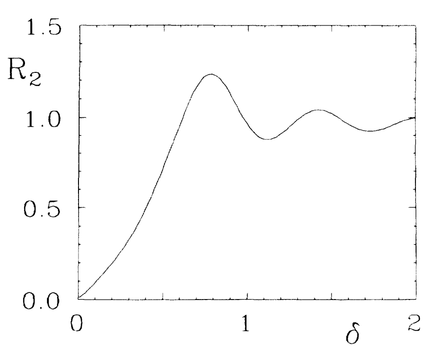

FIG. 1. Two-point correlation function R_2 of the roots lying on $\mathcal{C}$ for the SI polynomial (5) and $N=47$.

of classically chaotic systems and found that, as opposed to eigenstates of integrable systems, in the semiclassical limit $N\to\infty$ the roots tend to fill the whole phase space in a more or less uniform way. This ergodicity manifested by the distribution of roots of eigenstates of chaotic systems was interpreted as a quantum signature of chaos.

In order to further investigate this problem, we consider the distribution of the roots of Eq. (6) when the complex coefficients $\{a_k\}$ have a GRI distribution with the same standard deviation σ [15]. As discussed before, since the second moments are now large, the roots of $P(z)$ will spread over the complex z plane (liquid phase). The question is whether, at least in the semiclassical limit, the ergodic behavior of the roots is obtained. Using Kac's method, we prove that *for all N* the distribution of roots is *uniform* over the Riemann sphere,

$$\langle\rho(z)\rangle d^2z=\frac{N}{\pi}\frac{d^2z}{(1+|z|^2)^2}=\frac{N}{4\pi}\sin\theta\,d\theta\,d\phi\,. \qquad (7)$$

In proving this theorem, the binomial coefficients in Eq. (6) play a crucial role and a similar result cannot be obtained with other factors. To check the sensitivity of this result when the distribution of the coefficients is changed, we also consider a uniform distribution of coefficients. Although in this case Eq. (7) does not hold for finite N, it is recovered in the large-N limit.

(6) As was shown in part (4), if the roots of a polynomial are symmetric with respect to a certain line due to the existence of some functional equation, then the probability of having roots lying on that line is greatly enhanced. The influence of a symmetry in the distribution of roots is a general phenomenon. Two examples are provided by the concentration of roots on the real axis in the case of random polynomials with real coefficients [12], and the concentration on $\mathcal{C}$ of roots of SI polynomials. An extreme case is the Riemann zeta function.

To further illustrate these ideas, we consider a kicked-spin model quantum mechanically described by the one-step evolution operator [10,16] $U=\exp(-i\mu S_x)\times\exp(-ipS_z^2/2)$. The stationary equation $U|f_a\rangle=\exp(i\omega_a)|f_a\rangle$ determines the eigenphases ω_a and eigenstates $|f_a\rangle$ of U. The operator U commutes with two antiunitary operators [16]

$$T_1=e^{i\pi S_z}e^{i\mu S_x}K,\quad T_2=e^{-i\mu S_x}e^{i\pi S_y}K\,,$$

where K is the usual antiunitary complex conjugation operator. They satisfy $T_1^2=T_2^2=1$ and the time-reversal property $T_1UT_1=T_2UT_2=U^{-1}$. These two symmetries are nongeneric in the sense that (i) they are not just the conjugation operator usually connected to time-reversal invariance and (ii) they depend on the parameter μ controlling, together with p, the dynamics of the system. As explained before, in the coherent-state representation every eigenstate can be written as in (6), the coefficients a_k being obtained from the stationary equation. Because of these symmetries, the polynomial $P_a(z)$ associated to

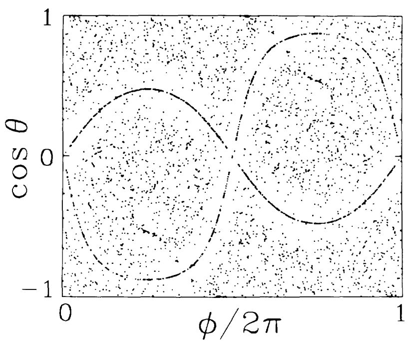

FIG. 2. The superposition of the roots of all eigenfunctions of the kicked-spin quantum map for $S=30$, $\mu=1$, $p=4\pi$.

$|f_a\rangle$ must satisfy $\overline{P_a(z)}=P_a(\mathcal{T}_j z)$, $j=1,2$, where the $\mathcal{T}_j$ are the classical versions of the quantum operators T_j. If z_p is a root of $P_a(z)$, then $\mathcal{T}_j z_p$ will also be a root. We therefore expect a strong concentration of roots of eigenstates of U on the symmetry lines $\mathcal{T}_j z=z$, $j=1,2$. In the stereographic variables (θ,ϕ), they take the form

$$\cos\theta=\pm\sin\phi\Big/\left(\sin^2\phi+\frac{1\mp\cos\mu}{1\pm\cos\mu}\right)^{1/2}, \qquad (8)$$

the upper and lower sign holding for the T_1 and T_2 symmetry, respectively.

Figure 2 shows the superimposition of the 60 roots of the 61 eigenstates obtained numerically for $S=30$, $\mu=1$, and $p=4\pi$, which classically corresponds to a fully chaotic dynamics [10]. We observe the expected concentration of roots on the two symmetry lines (8), a free-of-roots region close to them, and a tendency to cover in a more or less uniform way the remaining phase space.

An extra term in the propagator, like $\exp(ikS_y^2)$, breaks both antiunitary symmetries and the strong concentration of roots on the symmetry lines disappears. Now we obtain an approximately uniform distribution of roots over the phase-space sphere, in agreement with the results of section (5). Another manifestation of such T-symmetry breaking effect is the GOE → GUE transition in random-matrix theories [3,16].

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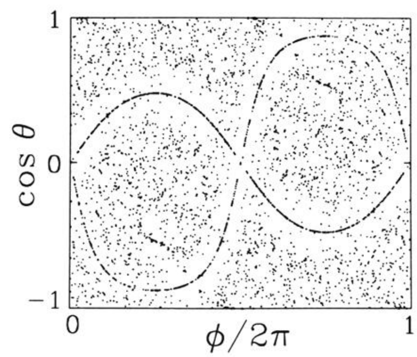

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