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Shapiro Steps in the Steady-State Dynamics of Incommensurate Structures

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We consider the overdamped ac-driven dynamics of an incommensurate one-dimensional array of coupled oscillators in a spatially periodic substrate potential (Frenkel-Kontorova model) by means of molecular-dynamics simulations. The ac steady-state trajectories are described by a two-dimensional hull function which, as the strength of the substrate potential varies, undergoes a transition by the breaking of analyticity. This critical behavior is macroscopically observed as the appearance of Shapiro steps in the response function.

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In an effort to disentangle the complexity of macroscopic systems with competing interactions, a great deal of attention has been paid in the last decade to simple many-body models (but complex enough to capture the essential physics); among these, the Frenkel-Kontorova (FK) type of model has been perhaps the most studied. Quite an important set of results on this class of models have provided deep insights into the equilibrium properties associated with the ubiquitous phenomenon of commensurability and its effects [1–3]. Of remarkable importance from a theoretical point of view is the fact that some of the exact as well as the high-precision numerical results concern the nonperturbative regime, where new concepts like the Aubry transition [4] play a central role. This transition is experienced by a static incommensurate (IC) structure when the (analytical) pinning potential grows above a certain critical value [which depends on the (in)commensurability ratio]. It is also intimately related to fundamental issues in the theory of dynamical systems (breakup of Kol'mogorov-Arnol'd-Moser tori). Its physical consequences are, among others, the pinning of the IC structure and its defectibility; below the Aubry transition the equilibrium IC structure is sliding and does not admit defects.

The dissipative dynamics of the FK model submitted to *dc forces* is well known to be of relevance to some interesting problems to which much experimental and theoretical work is being devoted: density wave transport (either charge- or spin-density wave) [5,6], Josephson-junction arrays [7], and flux line motion in layered type-II superconductors [8]. In all these systems, (in)commensurability effects and pinning (either due to impurity concentration or intrinsic) are widely recognized to be essential ingredients of an adequate description of the observed phenomenology: threshold depinning field (which depends on the commensurability ratio), nonlinear and oscillating (noisy) response, etc. Moreover, when these systems are submitted to *ac forces,* they show a steplike macroscopic response, often referred to as *Shapiro steps* (from the Josephson-junction language), or dynamical mode locking. Then, it seems rather natural to study the ac-driven dissipative dynamics of IC structures of the FK type of models, as we do here, in an attempt to gain insight into this common phenomenology.

On doing so, we show that the ac-driven dissipative dynamics of an *incommensurate* structure exhibits a transition between two distinct dynamical regimes. We characterize this transition as an Aubry transition, showing that a dynamical hull function describing the moving IC structure is analytical below the transition point, and nonanalytical above it. The physical consequence of this dynamical Aubry transition is the locking of the macroscopic response function, at certain resonant values. The *dynamical Aubry transition,* reported here for the first time, does not occur in the dc-driven dissipative dynamics of IC structures, and is the result of the competition of time scales in the ac-driven dynamics.

We study the dissipative (overdamped) [9] dynamics of an array of coupled harmonic oscillators u_j in a periodic substrate (pinning) potential V,

$$V(u)=[K/(2\pi)^2][1-\cos(2\pi u)] \tag{1}$$

(Frenkel-Kontorova model), submitted to dc and ac driving forces,

$$F(t)=\bar{F}+F_{\mathrm{ac}}\cos(2\pi\nu t)\,. \quad (2)$$

The equations of motion are

$$\dot{u}_j=u_{j+1}+u_{j-1}-2u_j-(K/2\pi)\sin(2\pi u_j)+F(t)\,, \quad (3)$$

where $j=-N/2,\ldots,N/2$; and we are interested in the thermodynamic limit $N\to\infty$. We use periodic boundary conditions and fix the interparticle average distance $\omega=\langle(u_{j+1}-u_j)\rangle$ to the desired (rational) value. As usual, irrational values of ω (IC structures) are approximated by their best rational approximants. Our choice of irrational here is the inverse golden mean, $\omega=(\sqrt{5}-1)/2$, that is best approximated by ratios of successive Fibonacci numbers: ... 21,34,55,89, ... ($\frac{34}{55}$ and $\frac{55}{89}$ were mostly used with no observed differences in the numerical results).

An important symmetry of Eqs. (3) is the following: Given a steady-state solution $\{u_j(t)\}$, the transformation $\sigma_{r,m,s}$,

$$\sigma_{r,m,s}\{u_j(t)\}=\{u_{j+r}(t-s/\nu)+m\}=\{u_j'(t)\} \quad (4)$$

(r,m,s arbitrary integers), produces another steady-state solution. Solutions that are invariant under this symmetry transformation will be called resonant trajectories.

In the *integrable* limit ($K=0$) the steady-state solution is simply given by

$$u_j(t)=j\omega+\bar{v}t+\alpha+(F_{\mathrm{ac}}/2\pi\nu)\sin(2\pi\nu t)\,, \quad (5)$$

where the (particle and time) average velocity $\bar{v}=\bar{F}$, and α is an arbitrary phase.

The behavior in the dc limit ($F_{\mathrm{ac}}=0$) has been previously studied by several authors [10,11], in connection with charge-density-wave transport. For a given IC structure there is a critical value $K_c(\omega)$ of the potential strength K, below which the structure slides for any $\bar{F}\neq 0$. Above $K_c(\omega)$ there is a threshold value (depinning force) F_t: If $\bar{F}>F_t$ the structure moves, but remains pinned if $\bar{F}<F_t$. This Aubry transition from "sliding" to "pinned," which occurs at $K_c(\omega)$, was first found and rigorously proved in the Frenkel-Kontorova model, and has later shown up in many other models [12]. In the sliding regime the incommensurate static structure is described by an analytical hull function that develops (infinitely many) discontinuities at the Aubry transition. As we mention above, the physical consequences of these nonanalyticities of the static hull function are, among others, the pinning of the incommensurate structure and its defectibility. When $\bar{v}\neq 0$, irrespective of the commensurability ratio, the steady-state trajectories can be expressed in terms of a monotonous and continuous [10,11] one-dimensional hull function $f(x)$,

$$u_j(t)=f(j\omega+\bar{v}t+\alpha)\,, \quad (6)$$

with the property $f(x+1)=1+f(x)$, or equivalently, $f(x)=x+g(x)$, with $g(x+1)=g(x)$. In (6), α is an arbitrary phase. The macroscopic response function $\bar{v}(\bar{F})$ is a strictly increasing function above the threshold value. Notice that the continuity of the dc-dynamical hull function $f(x)$ is assured all the way due to the continuity of the particle trajectories $u_j(t)$, whatever the pinning strength value K, once the motion is set in.

For nonzero values of both F_{ac} and K, two time scales are present in the system: The time scale (ν^{-1}) imposed by the periodic driving force $F(t)$, and the time scale ($\bar{F}^{-1}$) set up by the substrate periodic potential V. Both compete to determine the average velocity value, $\bar{v}$, of the motion. The set of differential equations (3), for $\omega=\frac{34}{55}$ (or $\frac{55}{89}$), was numerically integrated using a fourth-order Runge-Kutta algorithm. The time step used was 0.02τ (τ is the unit of time) and a time interval, typically around 100τ, was used as a relaxation time to allow the system to reach the steady state. $F_{\mathrm{ac}}=0.2$ and $\nu=0.2$ were kept as fixed parameters; no qualitatively different results were observed when F_{ac} and/or ν were varied, though more comprehensive studies on this issue are on the way. K and $\bar{F}$ have been varied, with a very fine discretization (10^{-4}–10^{-5}) in $\bar{F}$. Figure 1 shows the $(K,\bar{F})$ phase diagram obtained. At low values of the potential strength K, the average velocity (see Fig. 2, curve a) is given simply by $\bar{v}\propto\bar{F}$ (as at $K=0$). The moving incommensurate structure in this regime, at any value of the velocity, can be expressed in terms of a smooth two-dimensional hull function $f(x,y)$,

$$u_j(t)=f(j\omega+\bar{v}t+\alpha,\nu t+\beta)\,, \quad (7)$$

with the properties $f(x+1,y+1)=f(x+1,y)=1$

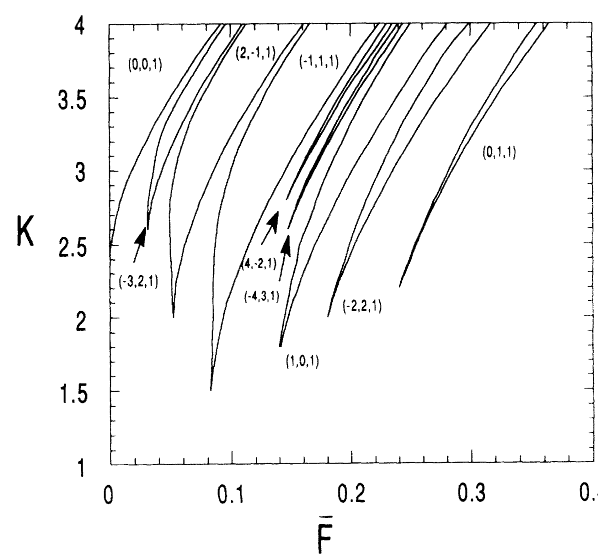

FIG. 1. Phase diagram of the different steady states in the $(K,\bar{F})$ plane, for $F_{\mathrm{ac}}=0.2$ and $\nu=0.2$. The triplets (r,m,s) stand for the integer values in Eq. (10) characterizing the resonant velocity value at which the locking occurs. For the sake of clarity, only tongues corresponding to a few resonant velocities are shown.

$+f(x,y)$, or equivalently, $f(x,y)=x+g(x,y)$, with $g(x,y)$ periodic in both variables; α and β are arbitrary phases. This smooth function [for an example see Fig. 3(a)] changes continuously with the potential strength K, approaching

$$f^0(x,y)=x+(F_{ac}/2\pi v)\sin(2\pi y) \tag{8}$$

as $K\to 0$ [see Eq. (5)]. In simple terms, the effect of the substrate potential at low K is a smooth modulation of the $K=0$ trajectory. This suggests that a perturbative approach to the functional equation satisfied by $g(x,y)$, i.e.,

$$\bar{v}[1+g_x(x,y)]+vg_y(x,y)=g(x+\omega,y)+g(x-\omega,y)-2g(x,y)-(K/2\pi)\sin\{2\pi[x+g(x,y)]\}+\bar{F}+F_{ac}\cos(2\pi y)\,, \tag{9}$$

should be plausible, provided ω is a "good" irrational [13] and the perturbation parameter K is small enough. In (9), $g_x(x,y)$ and $g_y(x,y)$ denote partial derivatives.

Above a certain value of K (that is around 1.5 for the particular values of ω, F_{ac}, and v that we are considering now) the variation of $\bar{v}(\bar{F})$ starts to develop "plateaus" at certain values of $\bar{v}$ (see Figs. 2, curves b and c). Those values are given by

$$\bar{v}/v=(r\omega+m)/s\,, \tag{10}$$

and the corresponding trajectories are invariant under the symmetry transformation $\sigma_{r,m,s}$ defined above in Eq. (4). As shown in Fig. 1, the particular value of K at which the plateau corresponding to a triplet (r,m,s) starts to develop does depend on these integers in a way that appears to be rather irregular. At each particular value $K_c(\bar{v})$, the corresponding two-dimensional hull function loses analyticity: Faults along the direction of vector $(\bar{v},v)$ in the x-y plane (physical time direction) are clearly developed [see Fig. 3(b)]. We are seeing an Aubry transition, whose macroscopic manifestation is the mode locking of the response function $\bar{v}(\bar{F})$, i.e., the appearance of Shapiro steps.

At high enough K, it seems that all the resonant values of $\bar{v}$ in a given interval have undergone the Aubry transition, and $\bar{v}(\bar{F})$ is presumably a devil's staircase (see Fig. 2, curve c) in that given interval. Of course, at very high values of $\bar{F}$ one leaves the tongues region in the phase diagram of Fig. 1, and no further steps show up there. The width $\Delta\bar{F}$ of the stability interval for a given resonant velocity, close to the Aubry transition value $K_c(\bar{v})$, scales in the following manner:

$$\Delta\bar{F}=A(K-K_c)^{\xi}\,. \tag{11}$$

Some examples of the fit by Eq. (11) for different resonances are shown in Fig. 4. The exponent ξ is not universal, but depends on the resonant value of the velocity; it could also depend on parameters like F_{ac} and v. For $F_{ac}=0$ and $\bar{v}=0$, the exponent should coincide with the scaling exponent for the depinning force in the static Aubry transition which is known to be [14] $\xi=3.0117,\ldots,$ while for $F_{ac}=0.2$, we have obtained numerically $\xi=1.6\pm0.05$.

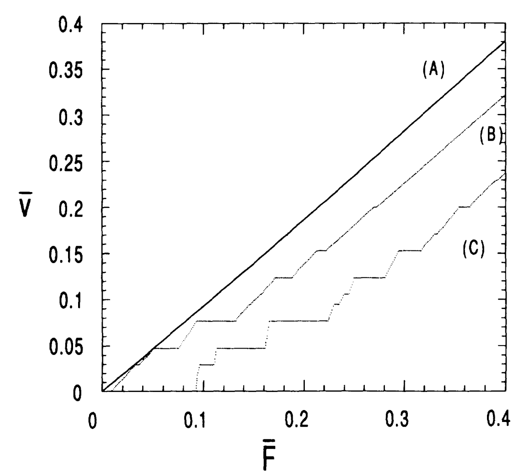

FIG. 2. Response function $\bar{v}(\bar{F})$, for $F_{ac}=0.2$ and $v=0.2$. Curve a, $K=1.4$ [just below the lowest value of $K_c(\bar{v})$]; curve b, $K=2.8$; curve c, $K=4$.

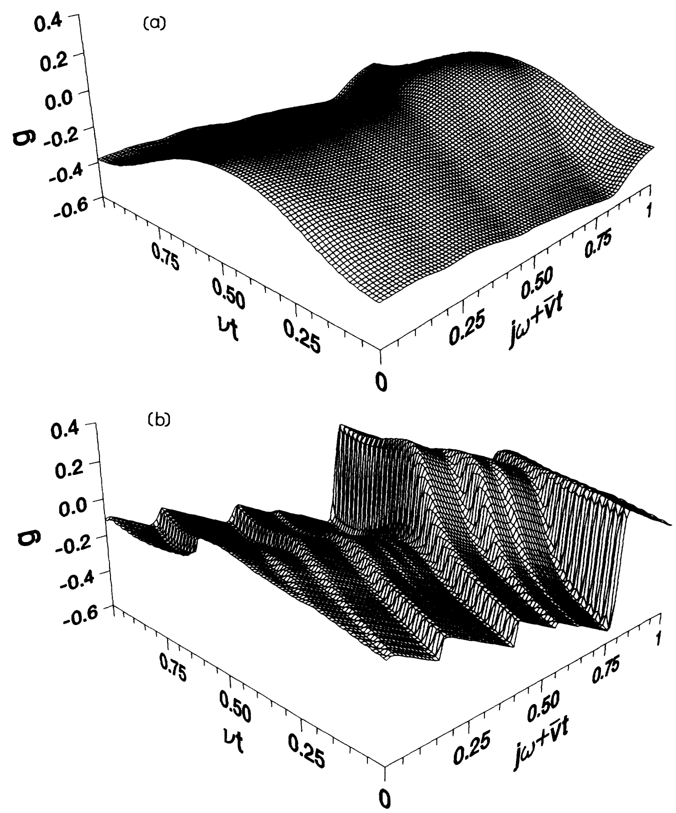

FIG. 3. Two-dimensional hull function $g(x,y)$ describing the motion of the incommensurate structure [see Eq. (9)] for the velocity (10) with $r=-1$, $m=1$, and $s=1$. (a) $K=1.4$; (b) $K=4$.

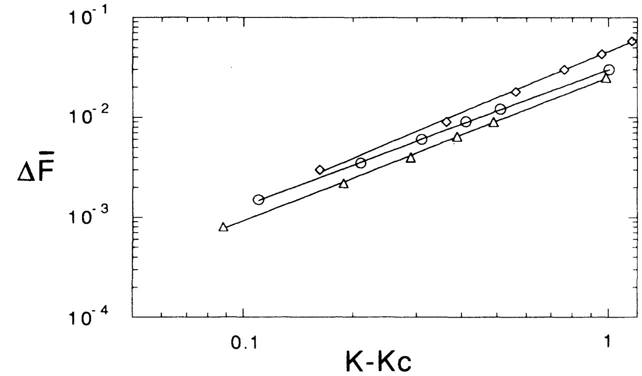

FIG. 4. Log-log plot of the fit by Eq. (11) for the velocities corresponding to (△) $(r,m,s)=(-1,1,1)$, $K_c=1.5$, and $\xi=1.6$; (○) $(r,m,s)=(2,-1,1)$, $K_c=1.9$, and $\xi=1.53$; and (◇) $(r,m,s)=(0,0,1)$, $K_c=2.4$, and $\xi=1.6$.

In summary, the ac-driven dissipative dynamics of incommensurate structures in the Frenkel-Kontorova model shows two regimes. At low pinning potential, the steady-state trajectories are described by a two-dimensional analytical hull function, and the response function $\bar{v}(\bar{F})$ is smooth and invertible. For each resonant value of $\bar{v}$ there exists a critical value of the potential strength above which the hull function is not analytical anymore, and the response function shows mode-locking behavior there. The transition between both dynamical regimes is an Aubry transition. The characterization of this transition through the critical behavior of different quantities is being studied, as well as the determination of its physical consequences. A more detailed study of this dynamical model including the commensurate case, the motion of discommensurations, and discussions on the unlocking mechanisms will be reported elsewhere.

We have argued that the dynamics of a simple model with competing interactions, such as the 1D FK model, is able to show the phenomenology observed in real systems. This model has some limitations: (a) First, we discuss the *dissipative limit* of the dynamics. This restriction could be plausibly satisfied by some of the physical systems mentioned in the introduction. The main effect of this restriction is the *effective reduction of degrees of freedom.* (b) The second limitation is the *one-dimensionality* of the model; a somewhat more restrictive condition, although symmetry considerations in some particular situations could give plausibility to a severe dimensionality reduction. (c) Last, we assume *convexity* of the interparticle interaction, which certainly could determine some important aspects of the conclusions, limiting its generality. On the other hand, the model is dealing with a discrete field, so that *intrinsic pinning* and not impurity pinning is considered.

Our numerical results for this model could stimulate both new theories and experiments in a number of physical systems involving incommensurate structures and both dc- and ac-driven forces. In particular, systems like incommensurate (charge or spin) density waves, or Josephson-junction arrays in the presence of a magnetic field, could be good candidates for the observation of the physical phenomena we describe in this specific model. The connection established here between the mode locking of the dynamical response function and the concept of an Aubry transition could pave the way to a satisfactory and unified understanding of the dynamical mode-locking phenomenology.

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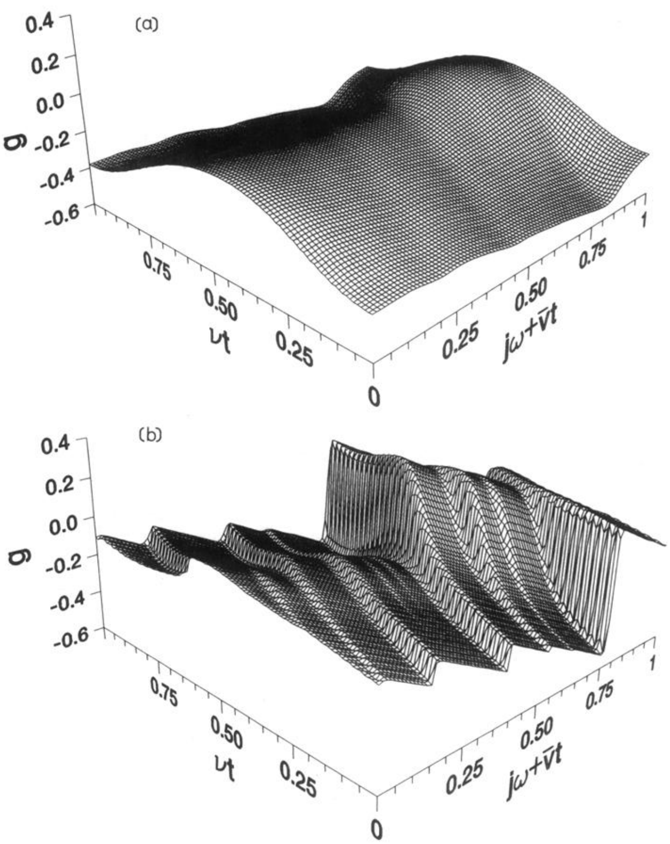

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