An Obstacle to Building a Time Machine

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Gott has shown that a spacetime with two infinite parallel cosmic strings passing each other with sufficient velocity contains closed timelike curves. We discuss an attempt to build such a time machine. Using the energy-momentum conservation laws in the equivalent (2+1)-dimensional theory, we explicitly construct the spacetime representing the decay of one gravitating particle into two. We find that there is never enough mass in an open universe to build the time machine from the products of decays of stationary particles. More generally, the Gott time machine cannot exist in any open (2+1)-dimensional universe for which the total momentum is timelike.

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It has long been known that general relativity allows the existence of closed timelike curves (CTC's) [1]. Since CTC's allow travel backwards in time, with attendant paradoxes, it is tempting to believe that they cannot arise in the real Universe [2]. Within the context of classical general relativity, one might therefore hope that the evolution of CTC's can be prevented by stipulating some reasonable constraints on the initial conditions.

Recently Gott [3] has shown that if two infinitely long straight cosmic strings pass each other with sufficient velocity, the resulting spacetime contains CTC's. This solution is particularly interesting because it requires no nontrivial spacetime topology, does not violate the weak energy condition, has no singularities or event horizons, and contains [4] complete spacelike hypersurfaces prior to which no CTC's exist. In this paper we explore the initial conditions under which Gott time machines can arise.

A spacetime populated solely by infinitely long parallel cosmic strings may be described by (2+1)-dimensional general relativity with point masses. Previous studies of (2+1)-dimensional gravity [5] have shown that the metric describing empty regions of space is necessarily flat. The external metric of a single particle with mass parameter μ is of the conical form: $ds^2=-dt^2+dr^2+r^2d\theta^2$, with θ in the range $0\le\theta\le 2\pi-\alpha$, where the deficit angle $\alpha=8\pi G\mu$ (G is Newton's constant). The spacetime can be constructed from Minkowski space by removing a pie slice of angle α and identifying the two sides of the slice.

Solutions with several moving particles may be constructed by joining boosted single-particle solutions. Such systems possess a conserved energy-momentum vector P^μ [6]. For Gott's spacetime we find that P^μ is tachyonic (i.e., spacelike); it is impossible to boost to a Lorentz frame in which the center of mass is at rest. This is equivalent to a previous result of Deser, Jackiw, and 't Hooft [7].

We use the laws of energy-momentum conservation to discuss nongeodesic motion, explicitly constructing the spacetime for one gravitating particle decaying into two. We then discuss an attempt to build a time machine from the decay of two stationary particles; the total deficit angle is found to exceed 2π, so the construction is impossible in an open universe. We state a more general conclusion about obstacles to time machine construction, which we will prove in a longer paper to follow.

Conservation laws in (2+1)-dimensional gravity.—In (2+1)-dimensional gravity with point masses, spacetime curvature is concentrated at the conical singularities of the particles. The curvature at these singularities can be probed by parallel transport of a vector through the spacetime—the flatness of the external space ensures that the result will be a topological invariant, depending only on the singularities encircled. For a stationary particle A with deficit angle α_A, the result of parallel transport of a vector counterclockwise is to transform it by a counterclockwise rotation matrix $R(\alpha_A)$. Parallel transport around a moving particle results in a transformation [6]

$$T_A=B_A R(\alpha_A) B_A^{-1}\,, \qquad (1)$$

where B_A is the Lorentz boost matrix that brings the rest vector to the velocity of particle A. A loop around several particles can be deformed to a sequence of loops that each encircle one particle; the resultant transformation is obtained by multiplication of the appropriate one-particle matrices: $T_{\rm tot}=T_N T_{N-1}\cdots T_1$. (Note that this definition requires a consistent ordering of the particles and a consistent definition of the coordinate systems in which the velocities of the particles are measured. This has been discussed in the literature [8,9], and will be discussed further in our longer paper.)

After a decay or scattering interaction, causality implies that the spacetime at large distances cannot be immediately affected, so parallel transport around a large enough loop must yield the same answer after the interaction as it did before. But the resulting matrix must be the same for all loops that enclose the particles, independent of the size of the loop. Thus, the Lorentz matrix $T_{\rm tot}$ that describes parallel transport around a group of interacting particles must be conserved; that is, evolution from a collection of N particles to M particles must satisfy

$$T_{\rm tot}=T_N T_{N-1}\cdots T_1 = T'_M T'_{M-1}\cdots T'_1 . \qquad (2)$$

This is the law of conservation of energy and momentum [6].

The three Lorentz generators $J\equiv M_{12}$ (rotation) and $K_i\equiv M_{i0}$ (boosts, with $i=1,2$) form a tensor $M_{\mu\nu}$, which can be represented by a pseudovector $\mathcal{J}_\mu=\frac{1}{2}\epsilon_\mu^{\lambda\sigma}M_{\lambda\sigma}$, where $\epsilon_{\mu\lambda\sigma}$ is the fully antisymmetric tensor ($\epsilon_{012}\equiv 1$), and indices are raised and lowered with the Lorentz metric $\eta_{\mu\nu}=\mathrm{diag}[-1,1,1]$. The matrix describing parallel transport can then be written as $T=\exp(-i\varphi^\mu\mathcal{J}_\mu)$, where φ^μ transforms as a Lorentz vector. For a single particle φ^μ is equal to $(8\pi GM,\mathbf{0})$ in the rest frame, so in an arbitrary frame it is $8\pi G$ times the standard energy-momentum vector $(\gamma M,\gamma M\mathbf{v})$. The total three-momentum P^μ of a group of particles is defined to be [6]

$$\exp(-i8\pi GP^\mu\mathcal{J}_\mu)\equiv T_{\rm tot}. \qquad (3)$$

In the limit $G\to 0$ the product (2) can be expanded as $T\simeq 1-8\pi iG\mathcal{J}_\mu\sum_n p_n^\mu$, so in this limit one recovers the standard conservation laws of special relativity. (Note that, for nonzero G, P^μ is generally not the sum of the individual p_n^μ.)

For convenience we represent the (2+1)-dimensional Lorentz group by 2×2 matrices [elements of SU(1,1)], taking $J=\frac{1}{2}\sigma_3$ and $K_i=\frac{1}{2}i\sigma_i$ (where the σ_i are the Pauli matrices). A rotation is given by

$$R(\alpha)=e^{-i\alpha J}=\begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2}\end{pmatrix}, \qquad (4a)$$

and a boost is

$$B(\xi)=e^{-i\xi\cdot\mathbf{K}}=\begin{pmatrix} \cosh(\xi/2) & e^{-i\phi}\sinh(\xi/2) \\ e^{i\phi}\sinh(\xi/2) & \cosh(\xi/2)\end{pmatrix}, \qquad (4b)$$

where ξ is the rapidity of the boost (related to the velocity by $\mathbf{v}=\hat{\xi}\tanh|\xi|$), and ϕ is the angle between ξ and the x axis. The matrix T for parallel transport around a particle is given by evaluating Eq. (1):

$$T_A=\begin{pmatrix} (1+p^2)^{1/2}e^{-i\alpha'/2} & ipe^{-i\phi} \\ -ipe^{i\phi} & (1+p^2)^{1/2}e^{i\alpha'/2}\end{pmatrix}, \qquad (5)$$

where

$$p\equiv\sinh\xi\sin(\alpha/2)=\gamma v\sin(\alpha/2) \qquad (6)$$

is a measure of the momentum of the particle, and α' is defined by

$$\tan(\alpha'/2)=\cosh\xi\tan(\alpha/2). \qquad (7)$$

Decay of one particle into two.—The decay of one self-gravitating particle into two is an intractable problem in (3+1)-dimensional gravity. In (2+1)-dimensional gravity the dynamical equations are simpler, but it is not obvious that there exists a consistent solution. The spacetime must be constructed by stitching together regions of flat spacetime in a way that maintains the proper deficit angle around each particle, but does not instantaneously change the spacetime at large distances from the decay. In this section we use the conservation laws to explicitly construct such a spacetime.

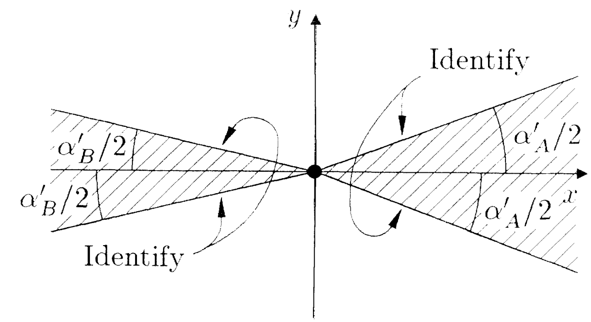

FIG. 1. An equal-time surface before the decay. For later convenience, the total deficit angle $\alpha_{\rm tot}$ is divided into two segments, α'_A and α'_B.

Consider the decay of a particle at rest, with deficit angle $\alpha_{\rm tot}$, into two particles A and B. The matrix for parallel transport around particle A and then B is given by $T^{BA}=T^BT^A$, with

$$T^{BA}_{11}=[(1+p_A^2)(1+p_B^2)]^{1/2}\exp[-i(\alpha'_A+\alpha'_B)/2]+p_Ap_B\exp[i(\phi_A-\phi_B)],$$
$$T^{BA}_{12}=i(p_A(1+p_B^2)^{1/2}\exp\{-i[(\alpha'_B/2)+\phi_A]\}+p_B(1+p_A^2)^{1/2}\exp\{i[(\alpha'_A/2)-\phi_B]\}), \qquad (8)$$
$$T^{BA}_{21}=T^{BA*}_{12},\quad T^{BA}_{22}=T^{BA*}_{11}.$$

The conservation law (2) requires that this matrix correspond to a rotation by $\alpha_{\rm tot}$ of the form of Eq. (4a). Equating these matrices yields the following relations [9]:

$$p_A=p_B\equiv p, \qquad (9)$$

$$\alpha'_A+\alpha'_B=\alpha_{\rm tot}, \qquad (10)$$

$$\phi_A-\phi_B=\pi-\alpha_{\rm tot}/2. \qquad (11)$$

In the $G\to 0$ limit, Eqs. (9) and (10) become conservation of momentum and energy in special relativity. Note that Eq. (11) indicates that the particles are not emitted at 180°; although they do come off back to back, the conical geometry implies that the angle between them is half of the total available angle, which is $2\pi-\alpha_{\rm tot}$.

It is useful to draw the constant-time surface before the decay so that the deficit angle $\alpha_{\rm tot}=\alpha'_A+\alpha'_B$ is divided into two pieces, as shown in Fig. 1. Consider the decay as witnessed by an observer in the upper half plane of the diagram, as shown in Fig. 2, with $\phi_A = \alpha'_A/2$ and $\phi_B = \pi - \alpha'_B/2$. The vertical velocity of particle A is given by $v_y = \tanh\xi_A \sin(\alpha'_A/2) = p_A/(p_A^2+1)^{1/2}$. Since $p_A = p_B$, the vertical velocity is the same for both particles, and the line that joins them is horizontal. Knowing that the metric far away is not affected by the decay, we can construct a coordinate system in the lower half plane so that the identification lines at large distances are the same as in Fig. 1. The spacetime is flat except at the locations of the particles, so the identification lines can be continued inward to the location of the moving particles. The particles are moving along the identification lines, so they can also be drawn in the lower half of the diagram. Since there is a unique straight line that joins A and B, the horizontal line in the upper half plane can be identified with the corresponding line in the lower half plane.

Two observers who were stationary relative to the particle before the decay, one in the upper and one in the lower half plane, will each see particles A and B moving toward them with a velocity component v_y. They will therefore have a relative velocity, as measured through the line connecting A to B, given by the Lorentz velocity addition formula $v = 2v_y/(1+v_y^2) = 2p(1+p^2)^{1/2}/(1+2p^2)$. (This is an example of the cosmic-string wake effect.) Equivalently, a particle which is at rest in the upper half plane of Fig. 2 will, when intercepted by the moving line AB, reappear in the lower half plane with a downward velocity v. A more complete description of the spacetime and its coordinatization will appear in our next paper.

The Gott time machine.—The time machine described by Gott [3] consists of two particles whose velocity vectors are at 180°, with nonzero impact parameter. This configuration contains closed timelike curves provided that each particle satisfies

$$g \equiv \cosh\xi \sin(\alpha/2) > 1 , \tag{12}$$

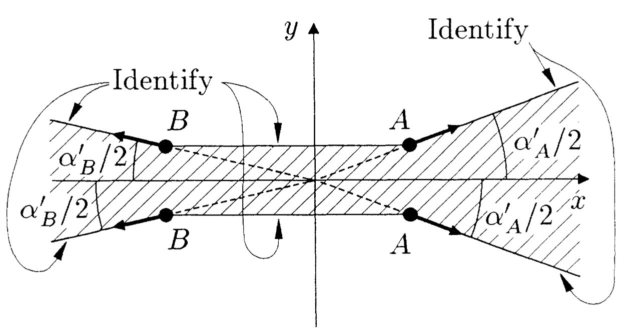

FIG. 2. An equal-time surface after the decay. The decay particles move along the identification lines, and at any time the two particles have the same y coordinate. Horizontal lines are drawn connecting the images of A and B both in the upper and lower half planes, and these two lines are identified. Note that the picture is consistent with causality: The spacetime far away is not affected by the decay.

where ξ is the rapidity of the particle, and α is its rest-frame deficit angle.

For simplicity we will consider the case of two identical particles. Parallel transport around a loop encircling the two particles results in a transformation matrix T^{BA} of the same form as Eq. (8), but with the parameters $\alpha_A = \alpha_B = \alpha$, $\xi_A = \xi_B = \xi$, $\phi_A = \pi$, and $\phi_B = 0$. To decide if this corresponds to a timelike or spacelike three-momentum, we need only calculate the trace of T^{BA}, since one can show that

$$\tfrac{1}{2}\,\mathrm{Tr}[\exp(-i8\pi G P^\mu \mathcal{J}_\mu)] = \begin{cases} \cos(4\pi G\sqrt{-P^2}) & \text{if } P^2 < 0, \\ \cosh(4\pi G\sqrt{P^2}) & \text{if } P^2 > 0. \end{cases} \tag{13}$$

From Eq. (8) one has $\frac{1}{2}\mathrm{Tr}T^{BA} = 1 - 2g^2$. Thus the requirement (12) implies that $\frac{1}{2}\mathrm{Tr}T^{BA} < -1$, which is not covered [10] by either of the cases in Eq. (13). However, SU(1,1) is a double cover of the Lorentz group SO(2,1), so the matrices $\pm T^{BA}$ correspond to the same Lorentz transformation. Since $\frac{1}{2}\mathrm{Tr}(-T^{BA}) > 1$, Eq. (13) implies that $P^2 > 0$; thus, the momentum of Gott's time machine is spacelike, or tachyonic.

To see the significance of the spacelike momentum, imagine approaching Gott's time machine, starting with slowly moving particles and then considering larger values of g. From the total momentum of the system as defined by Eq. (3), we can calculate the center-of-mass velocity:

$$v_{\mathrm{c.m.}} = \left(\frac{g^2 - \sin^2\alpha/2}{1 - \sin^2\alpha/2} \right)^{1/2} . \tag{14}$$

From this it is immediately obvious that $v_{\mathrm{c.m.}}$ approaches unity as $g \to 1$. Furthermore, the total deficit angle in the center-of-mass frame satisfies

$$\cos(\alpha_{\mathrm{c.m.}}/2) = 1 - 2g^2 , \tag{15}$$

implying that $\alpha_{\mathrm{c.m.}}$ approaches 2π as $g \to 1$.

The tachyonic nature of the three-momentum of the time machine does not imply that the solution is invalid—it remains an exact solution to the equations of general relativity. It does suggest, however, that the system might be impossible to construct in realistic situations. Suppose we try to build the time machine in a universe that starts without CTC's. For example, imagine two particles at rest, each of which decays into two particles. Each parent particle must then produce an offspring A that obeys the condition $g_A > 1$. To see the effect of such a large value of g, we can use Eqs. (6), (7), and (12) to express p_A in terms of g_A and α'_A. Using these equations again with $p_B = p_A$, we can express α'_B in terms of these variables and α_B. Equation (10) then implies that the deficit angle of the parent is given by

$$\alpha_{\mathrm{parent}} = \pi + \alpha'_A - 2\arcsin\{\sin(\alpha'_A/2)[g_A^{-1}\cos(\alpha_B/2)]\} . \tag{16}$$

This implies that if $g_A > 1$, then $\alpha_{\text{parent}} > \pi$. Since two parent particles are required to produce the time machine, the total deficit angle must exceed 2π, closing the (2+1)-dimensional universe—the decay of two stationary particles in an open universe can never create a Gott time machine.

The calculation of the previous paragraph can be generalized to the case of rocket propulsion, in which a particle emits many small ejecta. In our forthcoming paper, we demonstrate this and a stronger result: *In an open universe composed of point particles with a total momentum that is timelike, no two of these particles can ever have a high enough relative velocity to make a Gott time machine.*

We have found an insurmountable obstacle to building a time machine in (2+1)-dimensional gravity under a certain set of reasonable initial conditions. There is a sense in which, if it were possible to create CTC's, (2+1)-dimensional gravity would be the ideal arena—the absence of curvature in a vacuum prevents the creation of event horizons which could enclose the CTC's. Our inability to produce CTC's in this idealized situation suggests that there may also be obstacles to creating time machines with finite segments of cosmic string in the real (3+1)-dimensional world [11].

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Note added.—Since submitting this paper, we have shown that in a closed (2+1)-dimensional universe, it *is possible* to build a Gott time machine from the decays of initially static particles. The construction will be described in a forthcoming paper.

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[6] Deser, Jackiw, and 't Hooft [5] defined coordinate-identification matrices which they used to relate two moving particles to a single effective particle. The concept was pursued by a number of authors, including E. Witten, Nucl. Phys. **B311**, 46 (1988), and S. Carlip, Nucl. Phys. **B324**, 106 (1989). We are not aware of any papers that consider the application of the conservation laws to nongravitational processes.

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[9] After this paper was submitted we became aware of work by D. Lancaster and N. Sasakura [Classical Quantum Gravity **8**, 1481 (1991)], who also found our expressions (9) and (11), in the context of geodesic motion. Related work has been carried out by A. Cappelli, M. Ciafaloni, and P. Valtancoli, CERN Report No. CERN-TH-6093/91 (unpublished).

[10] Noncompact Lie groups may possess elements connected to the identity but not expressible as exponentials of elements of the Lie algebra. The SU(1,1) transformation matrix associated with Gott's time machine is in this category.

[11] This conclusion is reminiscent of Gott's discussion in Ref. [3]. See also Ref. [2].