Path-Integral Solution of the Telegrapher Equation: An Application to the Tunneling Time Determination

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Delay times relative to a beat envelope are deduced from a path-integral solution of the telegrapher equation, analytically continued to imaginary time when the square of the effective velocity becomes negative. The results are approximately described by a semiclassical model translated in frequency from the nominal to an effective cutoff, thus improving the agreement with experimental results.

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A path-integral treatment of the telegrapher equation, originally proposed by Kac [1], has been recently reconsidered in a very interesting paper by DeWitt-Morette and Foong [2] demonstrating the usefulness of the method. The essence of the procedure can be summarized as follows.

Let us consider the telegrapher equation

$$\frac{1}{v^2}\left(\frac{\partial^2 F}{\partial t^2}\right)+\frac{2a}{v^2}\left(\frac{\partial F}{\partial t}\right)-\frac{\partial^2 F}{\partial x^2}=0\,, \qquad (1)$$

where a is a positive constant and $F(x,0)=\phi(x)$, $(\partial F/\partial t)_{t=0}=0$, $\phi(x,t)$ being a solution of the wave equation without dissipation ($a=0$). The solution of Eq. (1) can be put in the form

$$F(x,t)=\tfrac{1}{2}\left[\langle\phi(x,S(t))\rangle+\langle\phi(x,-S(t))\rangle\right]\,, \qquad (2)$$

where $S(t)$ is a "randomized time" defined by

$$S(t)=\int_0^t(-1)^{N(\tau)}d\tau\,, \qquad (3)$$

$N(\tau)$ being a random variable with Poisson distribution of intensity a. The brackets in Eq. (2) denote averaging over all possible "checkerboard" paths connecting $x(0)=x_0$ and $x(t)=x_1$ [3].

It was demonstrated [2] that the solution $F(x,t)$ can be simply expressed by a quadrature:

$$F(x,t)=\int_{-\infty}^{\infty}\tfrac{1}{2}[\phi(x,r)+\phi(x,-r)]g(t,r)dr$$
$$=\int_0^{\infty}\tfrac{1}{2}[\phi(x,r)+\phi(x,-r)]h(t,r)dr\,, \qquad (4)$$

where $g(t,r)$ is the distribution of $S(t)$, while $h(t,r)\equiv g(t,r)+g(t,-r)$ is the distribution of $|S(t)|$. The functions $h(t,r)$ and $g(t,r)$ were evaluated by a Laplace-transform analysis and they are given in Refs. [2] and [4], respectively. The interest of this result lies in the fact that if we know a solution $\phi(x,t)$ of the wave equation without dissipation we can obtain the solution of the complete equation by evaluating the integrals in Eq. (4). In such a way $F(x,t)$ turns out to be constituted by the superposition of two contributions: one corresponding to a damped undistorted wave and the other, with a linear coefficient in a, to a distorted wave.

We shall apply this method to analyze the results of delay-time measurements in narrowed waveguides performed as a test of tunneling time models [5]. The results of this experiment achieved with a standard microwave set up in X band, in order to magnify the time range up to nanoseconds, are in rather good agreement with the predictions of quantum-mechanical models suitably translated into the (classical) electromagnetic framework. The transposition was relatively easy since the expressions of the delay time are directly related to the complex transmission amplitude which, in turn, is deduced from the (time-independent) Schrödinger equation for the motion of a particle of energy E in a potential V_0. This equation is formally identical to the Helmholtz equation for the propagation of a scalar field, electric or magnetic component of the wave, that is,

$$\frac{\partial^2\psi}{\partial x^2}+k^2\psi=0\,, \qquad (5)$$

where k is the wave number. The only difference is in the dispersive relation $k=k(\omega)$, which reflects the different time dependence in the Schrödinger equation with respect to the d'Alembert equation that describes the wave propagation without dissipation, Eq. (1) for $a=0$. Once the dispersion relations are properly taken into account, the results of quantum mechanics can be adopted for waveguides provided the substitution $\hbar/m\rightarrow c^2/2\pi\nu$ is made [5,6] (c is the light velocity). Even if the two wave equations describe the evolution of two quite different quantities—the quantum wave function and the electromagnetic field—this does not prevent doing a test of quantum-mechanical models, which are based on the evolution of wave packets, provided that the above substitution is made. There is, however, a limitation in this analogy since "in contrast to tunneling particle, an electromagnetic pulse can consist of many photons and can be probed in a noninvasive way" [7].

As for dissipative effects, which play a non-negligible role, a first attempt to include them was made by modeling a simple phenomenological scheme still obtained by analogy with quantum mechanics [6]. We maintain, however, that the best way to consider dissipation in wave propagation is to start from the telegrapher equation. This, in turn, could give a contribution towards the understanding of a complicated and controversial question, like that of tunneling time, when the difference in the dispersion relations is properly considered [8]. The analysis of Ref. [2] is performed for a nondispersive system, for instance a two-wire transmission line. There are therefore some difficulties in applying the above procedure [namely, Eq. (4) or, more explicitly, Eq. (24) in Ref. [2]] to a dispersive medium like a waveguide, which below the cutoff for the presence of evanescent waves can simulate quantum tunneling. Moreover, it is not immediately obvious how to apply Eq. (4) to any kind of signal.

These difficulties can be surmounted on the basis of the following assumptions: (i) We assume that the telegrapher equation holds for the propagation in a waveguide, and (ii) we apply Eq. (4) to a simple sinusoidal *progressive* wave of the type $\phi(x,t)=\sin(x-vt)$ corresponding to the beat envelope of the two waves (see below).

The first point is not trivial, as can be verified in the specialized literature [9,10], mainly for the presence of high-order modes induced by the narrowing of the waveguide. Assumption (i) holds if one considers only the principal mode *inside* the narrowed part of the waveguide.

The second point makes the application of Eq. (4) relatively simple [2,4]. At the same time the analytical results can be compared with the experimental ones from recent delay-time measurements, performed with two continuous waves of slightly different frequencies whose beat envelope was detected before and after the narrowing [5].

Let us consider the superposition of two waves of slightly different frequencies [11]:

$$\begin{aligned}\psi &= A\cos(\omega_1 t - k_1 x) + A\cos(\omega_2 t - k_2 x)\\ &= 2A\cos(\omega t - kx)\cos(\Delta\omega t - \Delta k x)\,, \end{aligned} \tag{6}$$

where $\omega_{1,2}=\omega\pm\Delta\omega$ and $k_{1,2}=k\pm\Delta k$. The first factor in the last member of Eq. (6) represents the carrier while the second one describes the beat envelope whose velocity $v_b=\Delta\omega/\Delta k$ is practically coincident with the group—or signal—velocity. Therefore, by considering v_b, we can predict the delay time versus frequency once the dispersion relation is known. Neglecting the influence of the carrier relative to the signal velocity—the Schuster model [11]—we can consider only the second factor in Eq. (6), namely, the beat envelope which, apart from unessential dephasing, can be written as a continuous wave of the type $\sin(x-vt)$, where v can be seen as the signal velocity. For our purposes, however, Eq. (4) cannot be applied directly since $\phi(x,r)+\phi(x,-r)$ does not represent a progressive wave but rather a stationary wave. A generalization of Eq. (4), with arbitrary mixing coefficients α and β of the progressive and regressive waves,

$$\alpha\phi(x,r)+\beta\phi(x,-r),\quad \alpha+\beta=1,\quad (\partial F/\partial t)_{t=0}\neq 0\,,$$

has been derived by Foong [4]. This allows us to obtain $F(x,t)$ for a progressive wave [$\sin(x-vt)$, $\alpha=1$, $\beta=0$] in the simple form

$$\begin{aligned}F(x,t) = e^{-at}[&\sin x\cos w_1 t - (v/w_1)\cos x\sin w_1 t\\ &+ (a/w_1)\sin x \sin w_1 t]\,,\end{aligned} \tag{7}$$

where $w_1=(v^2-a^2)^{1/2}$ is an effective velocity. The last term in Eq. (7) represents the "distorted wave" which disappears when $a\to 0$. The first and second terms correspond to the attenuated, nearly undistorted wave: In fact, for $a\ll v$, $w_1\simeq v$ and we have an attenuated progressive wave of the type $\sin(x-w_1 t)$ whose effective velocity w_1 is lower than v. So, according to Kac's predictions [1], the effect of dissipation is that of reducing the effective speed of the motion.

When $a>v$, the effective velocity w_1 turns out to be imaginary and $F(x,t)$ becomes an aperiodic function of the time, not suitable to interpret a delay-time experiment. We consider, as usually done in connection with semiclassical models, an analytic continuation of Eq. (7) for imaginary time ($it\to\tau$) [12]. In such a way we find that the nearly undistorted part resembles a pseudo progressive wave of the type $\sin(x-w_2\tau)$, where $w_2=(a^2-v^2)^{1/2}$. In this case the distorted wave has substantial influence on the amplitude $|F|$, but not on the delay time (see below).

Analogously, when v becomes imaginary ($v^2<0$), as in tunneling processes, the nearly undistorted part represents a pesudo progressive wave of the type $\sin(x-w_3\tau)$, where $w_3=(a^2+|v^2|)^{1/2}$. Note that in this case the effective (imaginary) velocity is increased by the dissipation [6].

So, on the basis of these considerations, we can derive a first approximate model of delay time which is simply given by $L/w_{1,2,3}$ (L is the length of the waveguide) in the different regions of v^2. In Fig. 1 the continuous line represents this model which is coincident with the semiclassical one apart from a significant shift of the singularity from $v^2=0$ to $v^2=a^2$.

More accurate results can be obtained by considering the complete form of $F(x,t)$ given by Eq. (7). The delay time is obtained as the difference in time between the same minima in the absolute value of $F(x,t)$, computed for $x=0$ and L, respectively. The results are reported in Fig. 1 by open circles and, for comparison, the results obtained by neglecting the last term in Eq. (7) (the distorted wave) are marked by crosses. We note that the inclusion of the distorted wave tends to *anticipate* the arrival of the signal in the tunneling region ($v^2<0$) while it tends to *retard* it in the classically allowed region ($v^2>a^2$). In the intermediate region ($0<v^2<a^2$) the two waves, which behave like stationary waves, give identical results. Something similar to the above results has

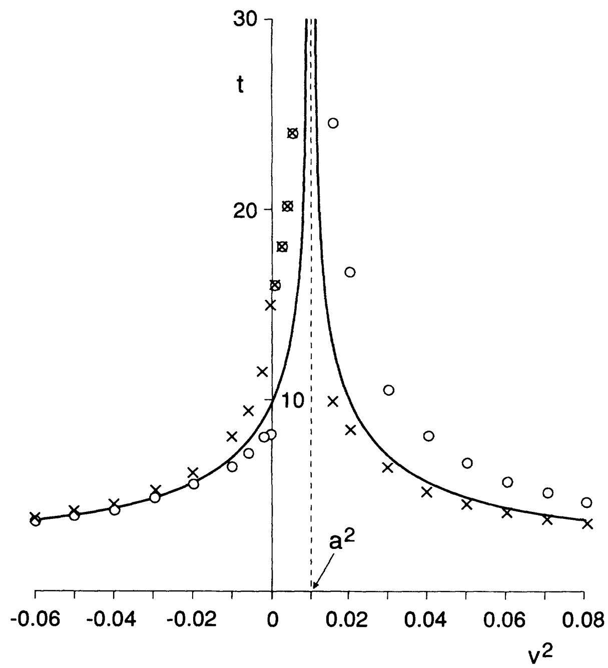

FIG. 1. Delay time (in arbitrary units) as a function of v^2 as deduced from Eq. (7) computed for $x=0$ and $x\equiv L=1$ and $a=0.1$. The continuous line corresponds to the simplified model $t=L/w_{1,2,3}$. The open circles are exact results for delay time while the crosses represent delay times obtained by neglecting the distorted wave, the last term in Eq. (7).

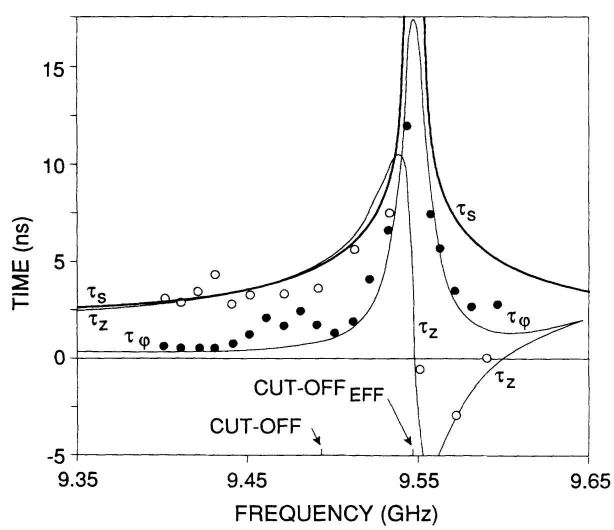

FIG. 2. Experimental results of τ_z vs frequency (open circles) as deduced from beat-envelope amplitude data for $L=15$ cm, compared with the relative theoretical curve and beat-delay data (solid circles) compared with the curve of τ_ϕ (see Ref. [5]). The heavy continuous line (τ_s) represents the modified semiclassical model resulting from the solution of the telegrapher equation with $a=0.1c$. This consists in a shift of the cutoff frequency from $\nu_0\simeq9.49$ GHz to $\tilde{\nu}_0\simeq9.54$ GHz.

been obtained in the analysis of a different type of signal, like a step function [13]. There, by taking into account the forerunner of the signal, the original semiclassical model turns out to be sensibly modified, resulting in a reasonable agreement with quantum-mechanical models and, more importantly, with experiments [5].

In order to compare the theoretical model derived here with the results of beat-delay measurements, we consider for simplicity the approximate result represented by the continuous line in Fig. 1. This can be easily compared with experiments by assuming, as anticipated earlier, that the signal velocity ($v\equiv v_b$) is coincident with the group velocity v_g in the waveguide.

The semiclassical delay time, in the absence of dissipation, is given by $t=L/|v_g|$, where v_g for the TE_{01} mode is [14]

$$v_g=c[1-(\lambda/2b)^2]^{1/2}. \tag{8}$$

Here λ is the free-space wavelength and b the width of the waveguide. In the presence of dissipation we have to consider the effective velocity $\tilde{v}_g\equiv w$ given as follows: (a) For $v>a$, that is, $\nu>(\nu_0^2+\delta^2)^{1/2}$, $\delta=a/\lambda$,

$$\tilde{v}_g=c\left[1-\left(\frac{\lambda}{2b}\right)^2-\frac{a^2}{c^2}\right]^{1/2}. \tag{9}$$

(b) For $v<a$ and $v^2<0$, that is, $\nu<(\nu_0^2+\delta^2)^{1/2}$,

$$\tilde{v}_g=c\left[\frac{a^2}{c^2}-1+\left(\frac{\lambda}{2b}\right)^2\right]^{1/2}. \tag{10}$$

In both cases the delay time is simply given by $L/\tilde{v}_g$. This means that the pure semiclassical model, with a singularity at $\nu=\nu_0$, turns out to be shifted from ν_0 to $\tilde{\nu}_0$, which can be interpreted as an effective cutoff given by $\tilde{\nu}_0=(\nu_0^2+\delta^2)^{1/2}$. The quantity δ can be determined by data fitting.

In Fig. 2 we report data for beat-envelope delay time (solid circles) relative to a narrowed waveguide of length $L=15$ cm and a cutoff frequency $\nu_0\simeq9.49$ GHz, and the curve of τ_ϕ (phase-time model) [5]. In the same graph we also report data for τ_z (open circles) and the relative theoretical curve. These data were experimentally determined according to the definition of τ_z in the electromagnetic framework:

$$\tau_z=\frac{1}{2\pi}\frac{\partial}{\partial\nu}(\ln T^{1/2}), \tag{11}$$

where T is the transmission coefficient [5]. The heavy continuous line represents the semiclassical model obtained here, which, because of dissipation, shows a shift in the cutoff frequency. The amount of this shift, as deduced from experimental behavior, is $\Delta\nu_0\simeq0.05$ GHz. This quantity corresponds to a value of the parameter a given by

$$a=\lambda\delta=\lambda(\tilde{\nu}_0^2-\nu_0^2)^{1/2}\simeq c(2\Delta\nu_0/\nu_0)^{1/2}\simeq0.1c,$$

which appears to be reasonable. As before, more accurate results for the delay time could be obtained by considering the complete form of Eq. (7), but the essential issue of the model is also captured by this simplified procedure. Thus, by inclusion of dissipation, we get a sensible improvement of the capability of the semiclassical model to describe the experimental results, even if the reported data are better described by the curves of τ_ϕ for the delay-time data, and τ_z for the corresponding points [15]. The latter appear to be in rather good agreement with the modified semiclassical model well below the effective cutoff, thus suggesting that the present treatment can also be connected to the Büttiker model [16] and relative transition-element analyses [17–19]. In fact, for opaque barriers, $T \simeq \exp(-2\kappa L)$, where κ is the inverse tunneling decay length. Thus, by Eq. (11) we have [5]

$$\tau_z = -\frac{2\pi v}{c^2\kappa}\frac{\partial}{\partial\kappa}(\ln T^{1/2}) \approx \frac{2\pi v}{c^2}\frac{L}{\kappa} \equiv \tau_s\,. \tag{12}$$

On the basis of the present work we can safely conclude that the path-integral treatment of the telegrapher equation, for the presence of dissipation, profoundly modifies the semiclassical model, making it a suitable candidate to interpret the experimental data. The results obtained here, together with those relative to a step-function signal [13], tend to supply a unifying scheme useful in overcoming the dichotomy inherent in the determination of the tunneling time [20].

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