PHYSICAL REVIEW LETTERS

VOLUME 68 | 20 JANUARY 1992 | NUMBER 3

Particle Dynamics in a Large-Amplitude Wave Packet

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(Received 22 May 1991)

A new adiabatic theory permits the understanding of one-dimensional dynamics of particles interacting with a large-amplitude wave packet for bounce time short compared with the transit time. This theory differs from previous ones in that the Hamiltonian varies slowly not with the time, but with the coordinate. The resulting adiabatic invariant is not equal, even in lowest order, to the usual action. This theory predicts the basic features of the interaction observed in previous numerical studies.

PACS numbers: 03.20.+i, 32.80.Pj, 52.20.Dq, 52.35.Ra

The interaction of particles with wave packets is central to plasma and accelerator physics. Nearly resonant particles, i.e., those with velocity close to the phase velocity of the wave, interact strongly with the wave. The large energy exchange between the wave packets and their associated resonant particles is important to plasma turbulence theory and crucial for charged-particle acceleration in slow-wave structures. At low amplitudes this interaction gives rise to the phenomena of Landau damping and quasilinear diffusion (cf. Ref. [1], Secs. 6.7 and 10.2), which can be analyzed perturbatively, assuming that the trajectories are only weakly modified from the straight unperturbed orbits. But at large amplitudes only numerical analyses, such as that of Ref. [2], are available. (A review of the literature is provided in Chap. 6 of Ref. [3].)

In this Letter we analyze particles interacting with large-amplitude wave packets, in which the bounce time is short compared with the time for a particle to cross the packet. For this problem, an adiabatic approximation should apply. However, as the Hamiltonian varies slowly not with time, but with the coordinate, the usual adiabatic theory does not apply, and the adiabatic invariant is *not* the familiar action, the phase-space area $\oint p\,dq$ enclosed by a contour of constant Hamiltonian at constant time. This result runs counter to the physical intuition firmly established in the plasma and accelerator physics communities.

We progress by reversing the roles of space and time, permissible in Hamiltonian theory. In the resulting system, the new coordinate is the rapidly varying phase, $\varphi \equiv \omega t - kq$, while the new time is the old coordinate q. Here the familiar adiabatic analysis applies, so the adiabatic invariant is the area in the phase plane of φ and its conjugate, enclosed by a contour of constant new Hamiltonian at constant new time. The analysis, then, consists largely of finding the new conjugate momentum and the new Hamiltonian.

This method applies to a wide variety of physical systems, such as the dynamics of electrons trapping and detrapping in strong Langmuir wave packets [2] or the ponderomotive potential of a free-electron laser [4], or ions trapping in a radio-frequency quadrupole [5]. Suitably modified, this method may be applied to electron dynamics in plasma beat-wave acceleration [6] or microwave plasma heating [7]. Outside the realm of charged-particle dynamics is the example of the evolution of high-frequency internal waves interacting with large-amplitude near-inertial wave packets in the ocean [8].

A specific physical situation is that of a large-amplitude wave packet in a plasma. Approaching the wave envelope, plasma electrons become trapped and are then carried along at the wave's phase velocity. The electrons detrap upon reaching the other side of the envelope where the wave amplitude decreases. The Hamiltonian for this system is

$$H(q,p,t) = p^2/2m + e\Phi$$
$$= p^2/2m + eA(q/L)\cos(kq - \omega t)\,. \qquad (1)$$

The amplitude is assumed to have the form $A(q/L) = A_0 f(q/L)$, where f is a function of unit peak and unit width, so that A_0 is the peak value of the potential, and L is the characteristic length over which the wave amplitude changes. This Hamiltonian has two dimensionless parameters. The first, $\varepsilon \equiv 1/kL$, is small for a wave packet, since its inverse is the number of wavelengths in the wave packet. The second quantity, $\alpha \equiv k^2 eA_0/m\omega^2$, is the ratio of the peak potential energy to the kinetic energy (in the stationary wave-packet frame chosen here) of a resonant particle.

Except for small $O(\varepsilon)$ differences, this Hamiltonian includes that studied by Fuchs, Krapchev, Ram, and Bers [2] (hereafter referred to as FKRB). The $O(\varepsilon)$ differences arise because they define the force to have the form of a slowly modulated sine. Our choosing the potential to have the form of a slowly modulated cosine adds to the force a small $O(\varepsilon)$ sine term. FKRB also chose the wave packet to have a specific Gaussian form.

Particles oscillate characteristically at the bounce frequency, $\omega_0 \equiv k(eA_0/m)^{1/2}$. The time for a resonant particle to cross the wave packet is kL/ω. Hence, the number of bounce oscillations that a resonant particle makes in crossing the wave packet is given by $\nu \equiv \omega_0 kL/\omega = \alpha^{1/2}/\varepsilon$. For small ν no bounce oscillations are completed. Here, as expected, FKRB found that quasilinear theory describes the dynamics well.

We analyze the large-ν regime, where adiabatic analysis is indicated, as the particle executes many bounce oscillations in crossing the wave packet. For adiabatic theory the Hamiltonian $H(q,p,\varepsilon t)$ must be a slow function of time. This is accomplished by using the quantities t and E, where $E=H$ is the value of the Hamiltonian, as the phase-space coordinates, since these quantities form [9] a canonical pair evolving according to Hamilton's equations with q acting as the time, when the Hamiltonian is taken to be the momentum as a function of the coordinate, time, and energy. That is,

$$\frac{dt}{dq} = \frac{\partial \wp}{\partial E} \text{ and } \frac{dE}{dq} = -\frac{\partial \wp}{\partial t},$$

where $\wp(q,E,t)$ is found by solving $H(q,p,t)=E$ for the momentum. For the system (1), the new Hamiltonian,

$$\wp = \pm [2mE - 2meA(q/L)\cos(kq-\omega t)]^{1/2}, \qquad (2)$$

has two branches: Where takes the plus sign, the new time q increases along trajectories, and where $\wp$ takes the minus sign, q decreases along trajectories.

The Hamiltonian (2) has its new temporal (q) variation not only in the slowly varying amplitude, but also in the rapidly varying phase. This rapid variation with respect to the new time q is eliminated by a canonical transformation making the phase $\varphi = \omega t - kq$ the new coordinate. The generating function $F(t,K,q) = (K + m\omega/2k^2)(\omega t - kq) - m\omega q/2k$, of the old coordinate t and the new momentum K, effects this transformation. The terms linear in the new momentum K ensure that the new coordinate conjugate to K is the phase $\varphi = \partial F/\partial K$. The remaining terms simply add constants to the new momentum and Hamiltonian that will aid in their interpretation. The standard relation, $E = \partial F/\partial t$, for the old momentum gives $K = E/\omega - m\omega/2k^2$. The new Hamiltonian,

$$P = \wp + \frac{\partial F}{\partial q} = -kK - \frac{m\omega}{k} \pm \left[2m\omega K + \left(\frac{m\omega}{k}\right)^2 - 2meA(q/L)\cos(\varphi)\right]^{1/2},$$

is in the form required for adiabatic theory: It varies rapidly with only the new coordinate φ, and it varies slowly with the new time q. Thus, the adiabatic invariant for the system is given by the loop integral, $J \equiv \oint K\, d\varphi/2\pi$, in the (K,φ) phase plane at constant new Hamiltonian P and new time q.

To aid the calculation of J, we interpret the transformed quantities. We first note that to integrate in φ at constant q is to integrate over the wave phase holding the wave amplitude fixed. This already differs from standard adiabatic theory, where the integration over the coordinate at fixed time would imply variation, though $O(\varepsilon)$, of the amplitude in the adiabatic invariant integral. Second, we note that $P = -kE_\varphi/\omega$, where

$$E_\varphi \equiv p_\varphi^2/2m + e\Phi \text{ and } p_\varphi = p - m\omega/k.$$

Thus, p_φ is the momentum in the *wave frame*, where the phase velocity vanishes, and E_φ is the energy in this frame. Hence, integrating at constant P is identical to integrating at constant wave-frame energy. Last, we note that the new momentum, $K = E_\varphi/\omega + p_\varphi/k$, essentially the wave-packet-frame energy, is a linear combination of the energy and momentum in the wave frame.

These facts allow us to evaluate the adiabatic invariant, $J = \oint d\varphi (E_\varphi/\omega + p_\varphi/k)/2\pi$, which must be done separately for the trapped particles (inside the separatrix) and the passing particles (outside). The separatrix is the contour $E_\varphi = eA$ or $P = -ekA/\omega$ of the new Hamiltonian in the K-φ plane. The first integral is $(E_\varphi/2\pi\omega)\oint d\varphi$, since E_φ is held constant in the integration. This piece vanishes for trapped particles, which complete a circuit in the phase. This piece is E_φ/ω for passing particles, for which φ increases by 2π in one period. The second term gives the usual action integral $J = \oint d\varphi\, p_\varphi/2\pi k$ for a wave of slowly varying amplitude. This action integral can be expressed in terms of complete elliptic integrals (see, e.g., Ref. [10] or p. 65 of Ref. [3]). So for passing particles,

$$J_p = E_\varphi/\omega + \bar{p}_\varphi/k, \qquad (3)$$

where $\bar{p}_\varphi \equiv \oint p_\varphi d\varphi/2\pi$, the wave-frame action integral, is the average momentum on a contour of constant wave-frame energy at fixed amplitude, given our choice in integrating in the positive-φ direction for passing particles moving slower or faster than the wave. For trapped particles the adiabatic invariant is an integral,

$$J_T = 2\bar{p}_\varphi/k \equiv (2/k)\int_{\varphi_L}^{\varphi_U} p_\varphi d\varphi/2\pi ,$$

between limits (φ_L, φ_U) of the bounce motion. For trapped particles, $\bar{p}_\varphi$ is not precisely an average momentum because (1) the range of the trapped phase is not 2π, and (2) the integration is effectively of $|\bar{p}_\varphi|$ because on the return path, when p_φ is negative, so is $d\varphi$.

The trajectories stay on contours of J, which are plotted in Fig. 1 in the plane formed by the average wave-frame momentum $\bar{p}_\varphi$ and the coordinate. The arrows on these contours indicate the direction of motion. The direction on the passing contours is determined at large distance, where the wave amplitude vanishes, and so the velocity is $\bar{p}_\varphi/m + \omega/k$. Trapped particles move to the right (with the phase) at constant $\bar{p}_\varphi$, the trapped-particle adiabatic invariant. The upper and lower branches of the separatrix $E_\varphi = eA$ are the two curves forming an outline of what resembles human lips. On these curves the average wave-frame momentum is $\bar{p}_{\varphi x} = \pm (4/\pi)(emA)^{1/2}$. The lower lip is shaded to indicate that this is an unphysical region: For trapped particles the quantity $\bar{p}_\varphi$ is positive by definition.

To illustrate the interpretation of this diagram we discuss the incoming trajectories c and e, which trap at the same amplitude: one at the resonant velocity minus the resonance width; the other at the resonant velocity plus the resonance width. Upon encountering the separatrix, the two classes of trajectories combine into a single trapped class. The trapped trajectories then travel through the wave packet. The trajectories detrap into the passing contours (now d and f) at the other end of the wave packet where the amplitude decreases. This is also what occurs when the amplitude varies only temporally.

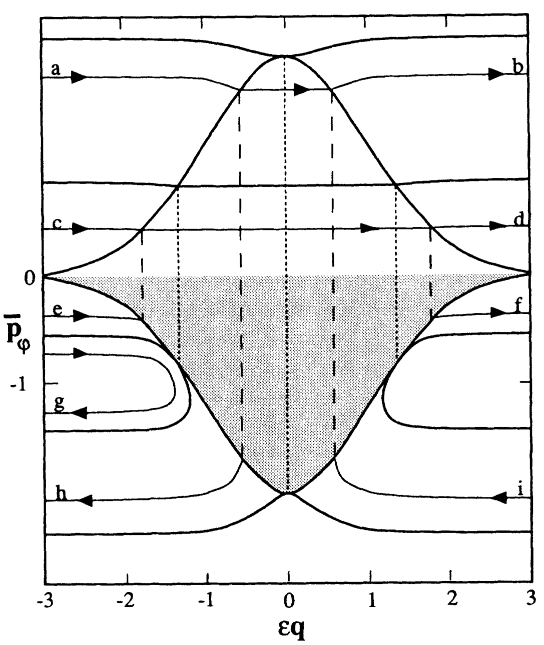

FIG. 1. Contours of the adiabatic invariant. The two curves resembling lips separate the passing particles (outside) from the trapped particles (inside). Typical trajectories are represented by curves with arrows. Heavier curves separate the different types of motion. For example, the heavier curve separating curve e from curve g is the boundary between trajectories (like e) that become trapped from those (like g) that reflect before encountering the separatrix.

However, our analysis also points to a new type of trajectory, such as that incoming on the contour labeled a. The slowing down (decrease of $\bar{p}_\varphi$) of this trajectory upon entering the wave packet is a manifestation of the ponderomotive force, which pushes particles away from regions of large oscillating field. If the phase is such that the trajectory becomes trapped, it encounters the separatrix again at the other side of the wave packet (where the amplitude decreases), and the trajectory leaves via contour b. The other possibility for an incoming trajectory on contour a is to encircle the trapped region and be reflected. This corresponds to jumping to the lower branch of the separatrix and exiting the region of the wave packet along contour h. This type of trajectory is present only when $\alpha > 2/\pi$.

The contour (b or h in this case) the trajectory ultimately leaves on is determined by the initial phase of the trajectory. The initial phase also determines a small $O(\varepsilon)$ change of the adiabatic invariant. This change would cause a phase-distributed set of trajectories to have an $O(\varepsilon)$ spread of values of the adiabatic invariant around that predicted by this lowest-order theory. Such separatrix crossing theory [11,12] has been applied to this situation elsewhere [3].

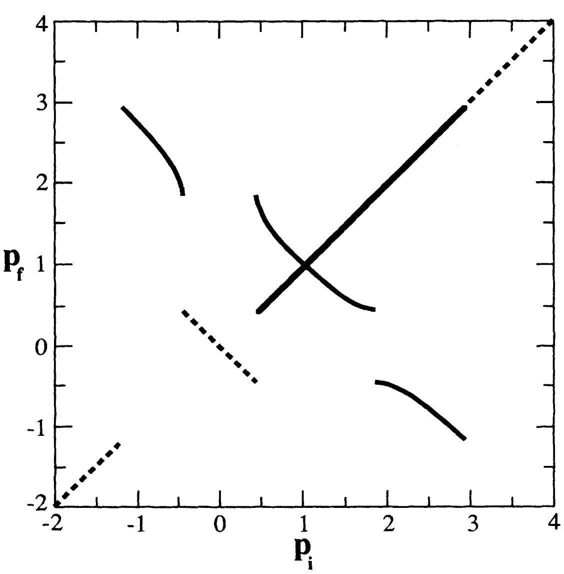

FIG. 2. Analytically calculated final momentum p_f after one interaction with the wave packet as a function of the initial momentum p_i for $\alpha = 2$.

Our analysis permits explicit calculation of the possible final values of momentum for a given value of incoming momentum. We illustrate this calculation for trajectory a. Far from the wave packet, a positive-p_φ particle has initial momentum p_i and, thus, adiabatic invariant $J_i = p_i^2/2m\omega - m\omega/2k^2$. This value of the adiabatic invariant is conserved up to the separatrix, which is encountered (to lowest order in the adiabaticity parameter) when J_i equals the separatrix value $(4/\pi k)(emA)^{1/2} + eA/\omega$ of the adiabatic invariant for a positive-p_φ particle. Therefore, the amplitude of encounter is given by $(emA)^{1/2} = [m\omega J_i + 4(m\omega/\pi k)^2]^{1/2} - 2m\omega/\pi k$. If the trajectory becomes trapped, it eventually detraps with positive p_φ and ultimately has final momentum equal to its initial momentum, as can be seen in Fig. 1. For a reflecting trajectory, the value of the adiabatic invariant changes to that, $J_f = -(4/\pi k)(emA)^{1/2} + eA/\omega$, of a negative-$p_\varphi$ particle at the separatrix. The trajectory then leaves the wave-packet region preserving the adiabatic invariant, so that far from the wave packet its momentum satisfies $J_f = p_f^2/2m\omega - m\omega/2k^2$. This chain of equalities gives the final momentum,

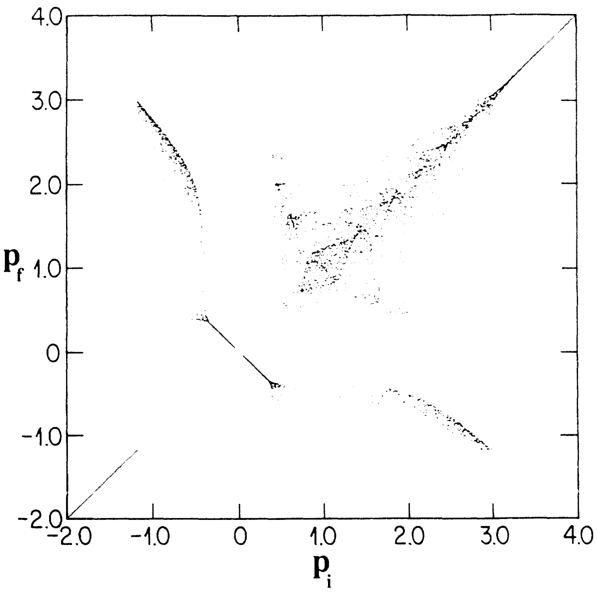

FIG. 3. Numerical results adapted from Ref. [2] for the final momentum as a function of the initial momentum.

$$p_f = -[p_i^2 + 32(m\omega/k\pi)^2(1 - \{1 + (\pi^2/8)[(kp_i/m\omega)^2 - 1]\}^{1/2})]^{1/2},$$

on contour h as a function of the initial momentum on contour a.

The collection of such considerations yields Fig. 2, a plot of the possible final momenta as a function of the initial momenta. Dashed curves are used when the associated trajectories do not intersect the separatrix, and so the adiabatic approximation does not break down. To illustrate the validity of our model, we compare Fig. 2 with Fig. 3, which shows the scattering results (p_f vs p_i for a distribution of initial phases) of Ref. [2] *previously obtained by purely numerical means.* Our results accurately provide the skeleton of the scatter plot. The spread around this skeleton is due to separatrix-crossing effects noted earlier. When there is no interaction with the separatrix, the skeleton is precise.

Our analysis, therefore, provides a basic analytic understanding of the interaction of particle trajectories in a spatially adiabatic wave. We have found a strong deviation from the temporally adiabatic case when the dimensionless amplitude is sufficiently large. These results are clearly important for the understanding of plasma turbulence consisting of large-amplitude wave packets and for the trapping and detrapping of particles in spatially varying accelerating structures.

We would like to acknowledge the helpful comments of Scott Hendrickson. This work was supported under U.S. Department of Energy Grant No. DE-FG02-86ER40302. The San Diego Supercomputer Center provided computational support.

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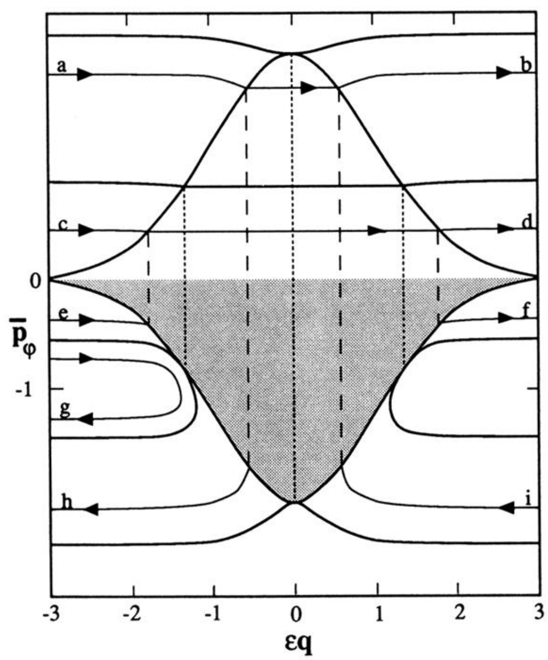

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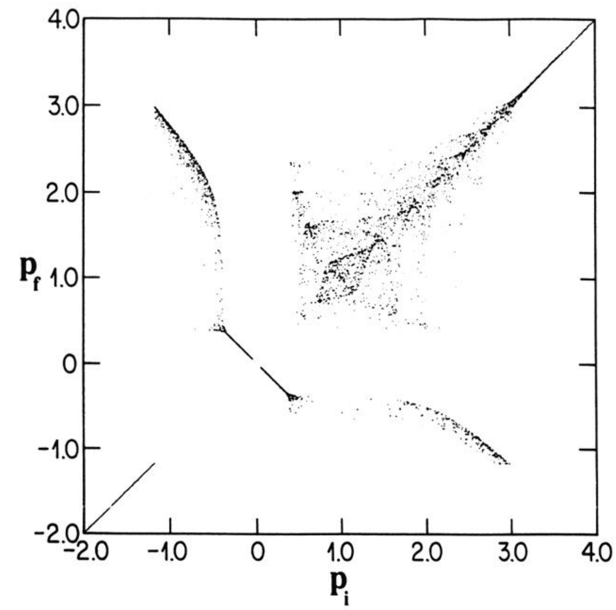

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