Deflagration Instability in the Quark-Hadron Phase Transition

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With the aim of determining the scale of inhomogeneities produced by the transition of quarks to hadrons in the early Universe, we study the hydrodynamic stability of slow combustion (deflagration). For a front velocity v, the phase boundary is unstable on a time scale $\tau \sim (1/v^3)$ fm; surface tension stabilizes bubbles below a critical size. For supercoolings implied by the bubble separations of interest for inhomogeneous nucleosynthesis (~ 1 m), τ is much less than the duration of the phase transition. Bubble disruption could restore homogeneous nucleosynthesis.

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First-order phase transitions, such as in the early Universe or the collision volume of an ultrarelativistic heavy ion collision, evolve dynamically. A region of the new phase is nucleated, which then grows as a bubble in the old phase. Such a phase transition likely occurs in the early Universe about 10 μs after the big bang, as hadrons (primarily pions) are formed from the primeval quark-gluon plasma. The conditions arising as bubbles propagate, coalesce and collide, and finally disappear, may have led to interesting cosmological consequences. Particularly intriguing is the possibility of baryon concentrations emerging from the phase transition [1], and their effects on subsequent nucleosynthesis [2,3]. In confrontation with observed elemental abundances, the products of nucleosynthesis in an inhomogeneous universe are consistent with larger average baryon densities than those of a homogeneous universe. A crucial question, then, is to what extent baryon concentrations can survive the dynamics of the phase transition.

The assumption of bulk phase separation is central to the idea of baryon concentration. It is usually assumed that hadron bubbles expand as spherical deflagration waves (subsonic condensation discontinuities) during the initial states of the phase transition [4,5], then undergo a process of collision and coalescence, and finally give way to collapsing spherical quark bubbles in a medium of hadrons. The actual situation could be very different if dynamic bubble growth is unstable. For some deflagration processes observed in the laboratory, for example, hydrodynamic instability leads to turbulent bubble growth [6]. A sufficiently potent instability would lead to efficient mixing of the phases, thereby washing out baryon concentrations, or preventing their occurrence altogether. Here we study the extent to which low-velocity deflagration in such a high-temperature gas as a quark-gluon plasma is hydrodynamically unstable.

For bubbles much smaller than a neutrino mean free path (~ 15 cm at 100 MeV), the latent heat evolved as quarks are converted to hadrons is efficiently transported by low-velocity bulk motions of the cosmic fluid. It is possible that radiative transport by neutrinos plays an important role for larger bubbles later in the phase transition [7]. Here we restrict our analysis to the hydrodynamic limit. On the scales of interest, the effects of gravity and universal expansion are negligible. We work in units $\hbar = c = k_B = 1$.

Fluid motions are described by the equations $\partial_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}$ is the stress tensor of an ideal relativistic gas, w is the enthalpy density, p is the pressure, and $u^\mu = \gamma(1,\mathbf{v})$ is the fluid four-velocity. For baryon-free matter the chemical potential is zero, and the thermodynamic quantities are related by $w = e + p = sT$, where e and s are the energy and entropy densities, T is the temperature, $dp = c_s^2 de$, and c_s is the sound speed. For small (but dynamically significant) supercooling, fluid velocities associated with the deflagration process will be small. For simplicity then, we consider only the low-velocity limit, i.e., $v \ll 1$. Taking the velocity and pressure as the basic variables undergoing perturbations, we have the linearized fluid equations for a low-velocity relativistic gas:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right] p' + wc_s^2 \nabla\cdot\mathbf{v}' = 0\,, \tag{1}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right] \mathbf{v}' + \frac{1}{w}\nabla p' = 0\,, \tag{2}$$

where primes denote perturbed quantities and quantities without primes describe the unperturbed states, assumed constant in time and space, except at the phase boundary.

For bubbles large compared to the phase-boundary thickness (~ 1 fm), we may treat the phase boundary as a discontinuity. We consider propagation of a planar interface that converts quarks into hadrons, and study perturbations of this surface. We work in the rest frame of the unperturbed surface, located at $z=0$, with quarks at $z<0$ and hadrons at $z>0$. The velocities v_q and v_h are the unperturbed flow velocities along the z axis. The general solutions of Eqs. (1) and (2) for the quark matter are

$$v'_{qz} = Ae^{i(kx-\omega t)}e^{q_q z}, \quad v'_{qx} = \frac{ik}{q_q}v'_{qz}, \quad p'_q = -\frac{w_q}{q_q}\tilde{k}_q v'_{qz}\,, \tag{3}$$

and for the hadronic matter are

$$v'_{hz} = e^{i(kx-\omega t)}[Be^{q_h z} + Ce^{i\omega z/v_h}] ,$$

$$v'_{hx} = e^{i(kx-\omega t)}\left[\frac{ik}{q_h}Be^{q_h z} - \frac{\omega}{kv_h}Ce^{i\omega z/v_h}\right] , \tag{4}$$

$$p'_h = -\frac{w_h\tilde{k}_h}{q_h}Be^{i(kx-\omega t)}e^{q_h z} .$$

Here ω is the complex frequency, k is a (real) wave number, and q_j and $\tilde{k}_j \equiv -i\omega + v_j q_j$ $(j=q,h)$ are both complex. The quantities A, B, and C are related by the boundary conditions. In the hadronic region, the particular solutions with the $e^{i\omega z/v_h}$ dependence correspond to $p'_h=0$. These particular solutions do not exist in the quark region since we are seeking solutions with $\mathrm{Im}\omega > 0$; such solutions grow without bound [8] for $z \to -\infty$. By substitution of these solutions into the fluid equations, we find the auxiliary conditions

$$\tilde{k}_j^2 + c_s^2(k^2 - q_j^2) = 0, \quad j = q,h , \tag{5}$$

with the requirements $\mathrm{Re}q_q > 0$ and $\mathrm{Re}q_h < 0$ to ensure boundedness of the solutions far from the front.

Boundary conditions are given by the conservation laws at the front. Continuing to treat the front as a discontinuity, we restrict our analysis to wavelengths long compared to the thickness of the front. For a planar discontinuity moving at constant velocity, the energy and momentum currents (T_{tz} and T_{zz}, respectively) are conserved across the front. For low velocities,

$$w_q v_q = w_h v_h \equiv F_H , \tag{6}$$

$$F_H v_q + p_q = F_H v_h + p_h , \tag{7}$$

where F_H is the energy flux, and velocities are with respect to the front.

The energy flux for a given amount of supercooling of the quarks is determined by the microscopic physical processes leading to hadronization at the interface. The maximum energy flow would occur if the interface behaved as a blackbody. For the low-velocity solutions we are concerned with, the exact form of the energy current will prove to be unimportant. Following Refs. [5] and [9] we take the energy flux to be proportional to the net blackbody energy flux between phases:

$$F_H = \alpha \tfrac{1}{4} g_h (\pi^2/30)(T_q^4 - T_h^4) , \tag{8}$$

where g_h is the effective number of helicity states in the hadron phase, $0 \le \alpha \le 1$, and $\alpha=1$ corresponds to the ideal situation. A similar prescription is used in classical bubble dynamics theory to constrain the net mass flux across a liquid-vapor interface [10].

The first boundary condition, from Eq. (7), gives linear perturbations in the pressure on each side of the interface,

$$\left[2\beta F_H\left(\frac{1}{w_q}-\frac{1}{w_h}\right) - F_H^2\frac{\theta}{w_q^2}+1\right]p'_q = \left[2\beta F_H\left(\frac{1}{w_q}-\frac{1}{w_h}\right)\eta - F_H^2\frac{\theta}{w_h^2}+1\right]p'_h - \sigma\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)\xi , \tag{9}$$

where $\eta \equiv g_q/g_h$ is the ratio of the effective number of helicity states between the two phases, $\beta \equiv 3\alpha/4\eta$, and $\theta \equiv 1 + 1/c_s^2$ ($=4$ for a relativistic gas). The final term represents the forces due to surface tension; σ is the surface mass-energy density of the interface, and ξ is the amplitude of small oscillations of the interface measured normal to the unperturbed interface. The spatial derivatives in ξ represent restoring forces of the distorted surface, while the time derivatives arise from the inertia of the interface.

The second and third boundary conditions, from Eqs. (6) and (8), give perturbations in the energy current on the two sides of the interface,

$$w_q\left(v'_{qz} - \frac{\partial\xi}{\partial t}\right) + \theta v_q p'_q = w_h\left(v'_{hz} - \frac{\partial\xi}{\partial t}\right) + \theta v_h p'_h = \beta(p'_q - \eta p'_h) . \tag{10}$$

The energy current is unaffected by the presence of constant surface tension [11].

The fourth boundary condition is obtained by requiring continuity in the transverse velocity across the interface:

$$v'_{qx} + v_q\frac{\partial\xi}{\partial x} = v'_{hx} + v_h\frac{\partial\xi}{\partial x} . \tag{11}$$

Perturbations of the front are described by $\xi = \xi_0 e^{i(kx-\omega t)}$. The four boundary conditions, Eqs. (9)–(11), give four equations in the five unknowns A, B, C, ξ_0, and ω; the indeterminance of one unknown, say, A, corresponds to the choice of overall scale. We obtain the dispersion relation

$$\frac{v_q}{v_h}[\beta(\eta-1)\tilde{k}_q - q_q][k^2 v_h(v_h - v_q) - \Omega^2] + \Omega\left[\beta(\eta-1)\tilde{k}_q\Omega + k^2 v_q - \frac{\tilde{k}_q}{\tilde{k}_h}(q_h + k^2 v_h)\right]$$
$$+ \frac{\sigma}{w_q}\frac{k^2+\Omega^2}{\tilde{k}_h}[\beta(\eta-1)\tilde{k}_h(q_q\Omega + k^2 v_q) - q_q(q_h\Omega + k^2 v_h)] , \tag{12}$$

where $\Omega \equiv -i\omega$, and we have neglected corrections to the coefficients of higher order in the velocities. Numerical solution shows that Eq. (12) has real roots Ω, such that $\Omega \ll k$. In order to obtain approximate analytical expressions, we seek solutions in this limit. In this regime, we see from Eq. (5) that $q_q \simeq k$ and $q_h \simeq -k$. We obtain the secular equation

$$\Omega^2(v_q+v_h)+2\Omega k v_q v_h+\left[k^2(v_q-v_h)+\frac{\sigma k^3}{F_H}\right]v_q v_h=0. \tag{13}$$

This equation is identical to the dispersion relation for small perturbations of a deflagration front propagating in a classical incompressible gas, with the enthalpy taking the role of the mass density [8]. Below a critical wave number k_c, one of the two roots of Eq. (13) is positive since $\mu \equiv v_h/v_q = w_q/w_h > 1$ (for $\mu < 1$, as in the conversion of hadrons to quarks, the front is always stable). Above $k_c = (\mu-1)w_q v_q^2/\sigma$, the system is stabilized by surface tension. The positive root Ω_+ grows monotonically with k for small k, and drops to zero at k_c. The instability grows over a time scale

$$\frac{1}{\tau}=\frac{\mu}{1+\mu}\left[-1+\left(1+\mu-\frac{1}{\mu}-\frac{1+\mu}{\mu}\frac{\sigma k}{w_q v_q^2}\right)^{1/2}\right]k v_q. \tag{14}$$

Since Ω_+ does not have an imaginary part, perturbations of the front are amplified without propagating along the front.

The evolution of the system is governed by the behavior of the mode with the shortest growth time scale. This mode is of wave number

$$k_0=\frac{2}{9}\frac{\mu}{1+\mu}\left[2+3\mu-\frac{3}{\mu}-\left(4+3\mu-\frac{3}{\mu}\right)^{1/2}\right]\frac{w_q}{\sigma}v_q^2. \tag{15}$$

Note the appearance of σ/w_q as the basic length scale.

We have in mind the following picture of the cosmological quark-hadron phase transition. The Universe supercools below a critical temperature and nucleates bubbles of hadrons which begin to grow at the expense of the quarks. Depending on the size of the nucleated bubbles, their initial growth is stable or unstable. If initially stable, the growing bubbles become unstable when they reach radii $\sim 1/k_c$; all modes with $k < k_c$ are unstable. Above a radius $1/k_0$, the most unstable mode begins to evolve. For the low degrees of supercooling considered here, $\mu \equiv w_g/w_h \simeq g_q/g_h \simeq 3$, and the most unstable mode grows on a time scale $\tau_0 \simeq 2.5\sigma/w_q v_q^3$, at a characteristic wave number $k_0 \simeq 1.1 w_q v_q^2/\sigma \simeq 0.55 k_c$.

The bubbles will be the least unstable if the phase transition occurs close to equilibrium. A quasiequilibrium phase transition lasts a time $t_H \sim 10^{-5}$ s. The extent to which the phase transition is dynamic is determined by the characteristic spacing of nucleated hadron bubbles, L. We estimate the average front velocity as $v_q \sim L/t_H$. The time scale for the instability is then

$$\tau_0 \simeq 2\times 10^{-13}\left(\frac{\sigma/w_q}{1\ \text{fm}}\right)\left(\frac{L}{1\ \text{m}}\right)^{-3}\ \text{s} \tag{16}$$

for bubbles larger than

$$R \sim \frac{1}{k_0} \simeq 10^{-6}\left(\frac{\sigma/w_q}{1\ \text{fm}}\right)\left(\frac{L}{1\ \text{m}}\right)^{-2}\ \text{cm}, \tag{17}$$

where a reasonable estimate for σ/w_q is ~ 1 fm.

For the instability to be dynamically relevant τ_0 must be less than the expansion time, and bubbles must be larger than $\simeq 1/k_0$ over this time scale. Both of these constraints are satisfied as long as $L/(1\ \text{cm}) \gtrsim 0.3(\sigma/w_q)$ fm^{-1}. For a spacing $L = 1$ m (100 m), the expanding bubbles become unstable on a time scale $\tau_0 \sim 2\times 10^{-13}$ s ($\sim 2\times 10^{-19}$ s) at radii above $\sim 10^{-6}$ cm ($\sim 10^{-10}$ cm). Typical values of L considered in calculations of inhomogeneous nucleosynthesis are 1–100 m [3]; for values of L much less than 1 m, diffusion of protons and neutrons after the phase transition destroys baryon concentrations before nucleosynthesis. However, we see that bubbles nucleated on a scale above ~ 1 m become unstable very early in the phase transition.

In obtaining these estimates, we have made the most conservative assumption of quasiadiabatic bubble growth. During the initial stages of the phase transition, however, before the released latent heat is thermalized and quasistatic expansion begins, front velocities could be considerably larger than later in the phase transition, and the instability correspondingly more potent. For example, with $v_q = 0.01c$ and $\sigma/w_q = 1$ fm, expanding bubbles would become unstable on a time scale of $\sim 10^{-17}$ s, at a radius $\sim 10^{-9}$ cm, even for a very small nucleation scale.

In summary, we have established the hydrodynamic instability of low-velocity deflagration in a relativistic gas, and have estimated the time scale of the instability. While the front is unstable to long-wavelength perturbations, the surface tension acts to stabilize the front at shorter wavelengths. Bubbles separated on scales of interest in calculations of inhomogeneous nucleosynthesis are unstable on time scales much less than the duration of the phase transition. The instability can begin to evolve only when a bubble reaches a size comparable to the unstable mode of shortest wavelength. For a bubble separation of 1 m, for example, bubbles larger than $\sim 10^{-6}$ cm are unstable on a time scale $\sim 10^{-13}$ s.

The consequences of the instability cannot be fully assessed within the linear analysis. The instability could be controlled by nonlinear effects, as in some cases of terrestrial combustion [12]. If nonlinear stabilization occurs in the early Universe, bubbles would grow approximately spherically. On the other hand, bubbles may disintegrate as they propagate, causing spherical bubbles to persist only below a certain size; the most extreme case would be the dissolution of the phase-separated medium into "foam." A sufficiently small bubble separation at the end of the phase transition would prevent baryon concentration effects, if they occur, from persisting until nucleosynthesis.

Freese and Adams [7] have suggested that if bubbles grow to a size comparable to a neutrino mean free path, radiative transport by neutrino conduction may dominate over hydrodynamic flow, and bubbles could become unstable to "dendritic growth." The hydrodynamic instability discussed here applies to smaller bubbles, and may prevent bubbles from becoming large enough for such a thermal instability to function.

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