Asymptotics of Reflectionless Potentials

I. Gjaja
Department of Physics, University of Maryland, College Park, Maryland 20742

A. Bhattacharjee[a]
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
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Analytic reflectionless potentials $\omega^2(\tau)$ are constructed for the one-dimensional equation $\epsilon^2 d^2q/d\tau^2 + \omega^2(\tau)q=0$. Unlike generic potentials which reflect waves with amplitudes of order $\exp(-1/\epsilon)$ as $\epsilon\to 0$, these potentials have reflection coefficients which are identically 0. It is shown that in the reflectionless case the adiabatic perturbation or iteration does not converge absolutely or terminate at some order. Since exact integrability is less restrictive than having a reflectionless potential, the case studied also shows that integrability does not imply convergence of the approximation methods used.

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The one-dimensional equation

$$\epsilon^2\frac{d^2q}{d\tau^2}+\omega^2(\tau)q=0\,, \tag{1}$$

with ω^2 real and positive on the real τ axis, analytic and nonzero in a strip about it, and approaching constant values as $\tau\to\pm\infty$, is applicable to a number of interesting physical problems. If q is a wave function, Eq. (1) can be recognized as the one-dimensional time-independent Schrödinger's equation for the above-barrier reflection problem. The above is also the equation of motion for the one-dimensional harmonic oscillator with the Hamiltonian

$$H=\tfrac{1}{2}[p^2+\omega^2(\epsilon t)q^2]\,, \tag{2}$$

where p is the momentum canonically conjugate to the coordinate q, $\tau=\epsilon t$, and t denotes time. The behavior of solutions of Eq. (1) in the limit $\epsilon\to 0$ has been a subject of considerable research—it describes the semiclassical limit of the quantum-mechanical problem, as well as the adiabatic limit of the harmonic oscillator with a frequency varying slowly in time. The generic behavior for $\epsilon\to 0$ is well known: The reflection coefficient (or, in the case of the classical harmonic oscillator, the change in the adiabatic invariant from $-\infty$ to $+\infty$) varies as $\exp(-c/\epsilon)$, where c is a constant of order 1 [1].

A phenomenon which accompanies such behavior is a divergence of perturbation theory or other, iterative procedures [2-4]. We have recently demonstrated [5] the *universal* character of such divergences in Lie perturbation theory and in an adiabatic iteration method [3] for a broad class of Hamiltonians. The fact that divergences and nonadiabatic effects [which in the case of Eq. (1) involve the generation of reflected waves] occur together in the generic case has prompted the speculation that if for Eq. (1) a reflectionless potential could be found, then perhaps perturbation or iteration methods would converge or terminate at some order [3].

The first goal of this paper is to show how to construct a class of reflectionless potentials for Eq. (1). We also point out why conditions given previously are insufficient [3,6]. To one such potential we apply two distinct mathematical procedures—the first is Lie perturbation theory, and the second, a WKB-type iteration procedure [7]. We show that neither of the procedures converges absolutely, even though the potential is reflectionless. This conclusion precludes the possibility of making a general link between reflections and divergences. More generally, since the system given by Eq. (2) is exactly integrable even for a generic choice of $\omega^2(\tau)$, as shown below, we are able to conclude that integrability does not imply absolute convergence of these approximation methods. (By integrable we mean that in the phase space extended to include t and its conjugate momentum, the system has two independent first integrals in involution.)

Before we proceed with the construction of reflectionless potentials, we consider in some detail the problem of the classical harmonic oscillator for which the Hamiltonian is given by Eq. (2). This system has been studied extensively in Refs. [8-10]. In addition to the obvious invariant $H(q,p,t)-H$, there exists an additional invariant $J=\tfrac{1}{2}[\rho^{-2}q^2+(\rho p-\epsilon\dot{\rho}q)^2]$, where ρ is any solution of the equation

$$\epsilon^2\ddot{\rho}+\omega^2(\tau)\rho-1/\rho^3=0\,. \tag{3}$$

(Here we denote $d/d\tau$ by an overdot.) The connection between Eqs. (1) and (3) is seen by writing the exact solution to Eq. (1) in the form

$$q_{1,2}(\tau)=\rho(\tau)\exp\left[\pm\frac{i}{\epsilon}\int_{-\infty}^{\tau}d\tau'\rho^{-2}(\tau')\right]. \tag{4}$$

Substituting Eq. (4) into Eq. (1) and differentiating then gives Eq. (3). Conversely, as shown in Ref. [9], *any* two linearly independent solutions $q_{1,2}$ of Eq. (3) determine ρ,

$$\rho(\tau)=\pm(\epsilon\alpha)^{-1}\{A^2q_1^2+B^2q_2^2 \pm 2[A^2B^2-(\epsilon\alpha)^2]^{1/2}q_1q_2\}^{1/2}. \tag{5}$$

Here α is the constant Wronskian, A and B are two arbitrary complex constants, and we are free to choose any combination of signs.

We denote by S the strip about the real τ axis where $\omega^2(\tau)$ is nonzero and analytic, and for definiteness we set $\omega^2(\pm\infty)=1$. (Note that S is a simply connected domain. For the purposes of the next paragraph it is only necessary that ω^2 be analytic in S.) In order to insure the validity of the perturbation procedures to be applied, we take the changes in ω^2 to occur over an interval of t of the order of $1/\epsilon$ or less. This is a stronger requirement than $\int_{-\infty}^{+\infty}|\omega^2(t')-1|dt'<\infty$ which is needed for the proof we now give.

We turn to Eq. (3) and impose the (obviously permissible) requirement that as $\tau\to-\infty$, $\rho(\tau)=\omega^{-1/2}(\tau)=1$. Then a bounded, real, positive solution, analytic in some finite region near the real τ axis, must exist for Eq. (3). This can be demonstrated as follows: Given the conditions on ω^2 stated above, if we pick two linearly independent vectors at some $\tau_0\in S$ [consisting, for example, of $q(\tau_0)$ and $\dot q(\tau_0)$], Eq. (1) will have two linearly independent solutions $q_1(\tau)$ and $q_2(\tau)$ which satisfy the two sets of initial conditions at τ_0, and which are analytic for all $\tau\in S$. Furthermore, $q_1(\tau)$ and $q_2(\tau)$ are bounded for all real τ [11]. Since we can always choose the two sets of initial conditions to be real, and hence the solutions to be real on the real τ axis, all that remains to complete the proof is to show that $\rho(\tau)$ is never 0 for real τ. This is not hard. First, q_1 and q_2 cannot be simultaneously 0, for if they were, then $\alpha=0$ for all τ, and the solutions would not be linearly independent. So, let us suppose that at some $\tau=\tau_1$, ρ takes on the value 0. From Eq. (5) this requires $A^2q_1^2+B^2q_2^2=\mp2[A^2B^2-(\epsilon\alpha)^2]^{1/2}q_1q_2$. Upon squaring this expression we are left with $(A^2q_1^2-B^2q_2^2)^2=-4(\epsilon\alpha)^2q_1^2q_2^2<0$, which is a contradiction, since we can always choose q_1, q_2, their derivatives, and hence α real. Therefore, ρ which possesses all the properties stated above exists. Thus the invariant J exists, and the Hamiltonian system is exactly integrable (regardless of the value of ϵ). By a canonical transformation to the action-angle variables where J plays the role of the action [9], the Hamiltonian can be brought into the form $K(J,\Phi)=\rho^{-2}J$.

We are now prepared to return to Eq. (1) and address the question of reflections. This problem is conveniently described in terms of the asymptotic behavior of $\rho(\tau)$ as $\tau\to+\infty$. By assumption, $\lim_{\tau\to-\infty}\rho(\tau)=1$, that is, as $\tau\to-\infty$ the solutions given by Eq. (4) represent plane waves. As shown in Ref. [10], then the asymptotic behavior of ρ as $\tau\to+\infty$ is

$$\rho(\tau)=[\cosh(\delta)\pm\sinh(\delta)\sin(2\tau+\phi)]\,. \tag{6}$$

Here δ and ϕ are real parameters. Their values, as well as the choice of sign, depend on the functional form of $\omega(\tau)$. [We remark here that is is easy to generalize our calculation to the case $\omega(-\infty)=K_1$, $\omega(+\infty)=K_2$, where K_1 and K_2 are positive real constants. We then take $\rho(-\infty)=K_1^{-1/2}$ and $\rho(+\infty)$ as given by Eq. (6) multiplied by $K_2^{-1/2}$.]

From Eq. (6) it is now clear that $q_{1,2}(\tau)$ as given by Eq. (4) do not, in general, represent plane waves as $\tau\to+\infty$. [This is most easily seen by noting that $|q_1(\tau)|^2$ and $|q_2(\tau)|^2$ are oscillatory functions of time, not constants.] It is therefore not sufficient for a reflectionless potential to simply require that the solution be of the form $q(\tau)=\rho(\tau)\exp[(i/\epsilon)\int_{-\infty}^{\tau}d\tau'\rho^{-2}(\tau')]$, as proposed in Refs. [3] and [6]. (The same flaw invalidates a proof given in Ref. [2].)

We give a prescription for the construction of reflectionless potentials: Choose any positive $\rho(\tau)$ which is analytic and nonzero in a strip about the real τ axis, which satisfies $\epsilon^2\rho^3\ddot\rho<1$ for all real τ, and for which $\lim_{\tau\to\pm\infty}\rho(\tau)=1$. (It is the last property that gives $\delta=0$ and insures that the solution $q(\tau)=\rho(\tau)\exp[(i/\epsilon)\times\int_{-\infty}^{\tau}d\tau'\rho^{-2}(\tau')]$ is a wave propagating purely in one direction at both $\tau\to-\infty$ and $\tau\to+\infty$.) Substitute this $\rho(\tau)$ into Eq. (3) and solve for $\omega^2(\tau)$ to get a reflectionless potential which is analytic in a strip about the real τ axis. For the harmonic oscillator (2), the same condition makes the change between the final and initial values of the adiabatic invariant identically zero.

We now consider an example of a reflectionless potential which is also well known in soliton theory [6,12]. We take $\rho(\tau)=[(1+\epsilon^2\tanh^2\tau)/(1+\epsilon^2)]^{1/2}$ which gives

$$\omega(\tau)=(1+2\epsilon^2\operatorname{sech}^2\tau)^{1/2}\,. \tag{7}$$

[For another example, less tractable analytically, see Appendix D of Ref. [5]. This example is interesting because $\rho(\tau)$ does not depend on ϵ.] From Eq. (7) it is evident that as $\tau\to\pm\infty$ we have $\rho(\tau)=\omega^{-1/2}(\tau)=1$. It is worth emphasizing here that whereas the Hamiltonian (2) is exactly integrable for a very wide class of functions $\omega(\tau)$, only a subset of these are reflectionless. Thus, both plots in Fig. 1 are numerical solutions of Eq. (3), with $\rho(-\infty)=1$ in both cases (for numerical purposes $-\infty$ is taken to be -100). The top plot describes $\rho(\tau)$ for a generic potential that generates reflections, $\omega^2(\tau)=(\tau^2+2)/(\tau^2+1)$. We have deliberately chosen ϵ to be large ($\epsilon=0.6$) for this curve, in order to make the oscillations for large τ clearly visible. The bottom plot, on the other hand, shows the solution $\rho(\tau)$ for $\omega(\tau)$ given by Eq. (7) (for $\epsilon=1$). We call attention to the striking difference in the asymptotic behavior of $\rho(\tau,\epsilon)$ for large τ in the generic and reflectionless cases.

We now apply Lie perturbation theory to the Hamiltonian (2), with $\omega(\tau)$ given by Eq. (7), and show that neither the fact that the system is integrable nor that the potential is reflectionless causes the perturbation expansion to converge absolutely. We rewrite the Hamiltonian in the form $H=\omega I+\epsilon I(\dot\omega/2\omega)\sin2\theta$, where I and θ are the action-angle variables for the case $\epsilon=0$. Expanding the explicit ϵ dependence (not multiplying t) to $O(\epsilon^3)$ gives

$$H=I+\epsilon^2I\operatorname{sech}^2\tau-\epsilon^3I\operatorname{sech}^2\tau\tanh\tau\sin2\theta+O(\epsilon^4)\,. \tag{8}$$

The Hamiltonian (8) is amenable to the Lie method which is described, for instance, in Ref. [13]. Here we follow the notation of Ref. [5] (where a brief description of the method is also provided). In our example, the generating function carries the expansion

$$w(I,\theta,\tau)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\epsilon^{n+k}w_{n+1,k}\,. \tag{9}$$

For the Hamiltonian (8), it can be easily shown that $w_{1k}=w_{2k}=0$ for all k. For $n=3$ we get

$$w_{3k}=(-1)^{k+1}\frac{3}{2^k}I\left(\frac{e^{2i\theta}}{i^k}+\frac{e^{-2i\theta}}{(-i)^k}\right)\frac{d^k}{d\tau^k}\frac{e^{\tau}-e^{-\tau}}{(e^{\tau}+e^{-\tau})^3}\,. \tag{10}$$

From Eq. (10) it can be seen that in the limit $k\to\infty$, $\epsilon^{2+k}w_{3k}$ also tends to infinity because of derivatives with respect to τ, even for arbitrarily small ϵ. For compactness of the formulas we demonstrate this fact at $\tau=0$; the extension to arbitrary τ is straightforward. It can be shown that at $\tau=0$ we have the identity

$$\left.\frac{d^k}{d\tau^k}\frac{e^{\tau}-e^{-\tau}}{(e^{\tau}+e^{-\tau})^3}\right|_{\tau=0}=\begin{cases}(-1)^{(k-1)/2}(k+2)!\dfrac{2^{k+1}}{\pi^{k+3}}\left(1-\dfrac{1}{2^{k+3}}\right)\zeta(k+3), & k\text{ odd}\,,\\ 0, & k\text{ even}\,,\end{cases} \tag{11}$$

where $\zeta(k)$ is the Riemann zeta function. For $k\to\infty$ we can replace $(1-2^{-k-3})\zeta(k+3)$ by 1. This then is in agreement with the Cauchy-Hadamard formula,

$$\limsup_{k\to\infty}\left(\left|\frac{1}{k!}\frac{\partial^k f(\tau_0)}{\partial\tau^k}\right|\right)^{1/k}=\frac{1}{R}\,,$$

where f is an arbitrary function and R is the distance from τ_0 to the nearest singularity of f, since the singularities of $(e^{\tau}-e^{-\tau})/(e^{\tau}+e^{-\tau})^3$ nearest to $\tau=0$ are at $\tau=\pm i\pi/2$. (The Cauchy-Hadamard formula can be used to obtain easily the dominant behavior of $d^k[(e^{\tau}-e^{-\tau})/(e^{\tau}+e^{-\tau})^3]/d\tau^k$ for any τ.) Hence,

$$\epsilon^{2+k}w_{3k}\to(-1)^{(k-1)/2}\frac{\epsilon^{2+k}(k+2)!6I}{\pi^{k+3}}\left(\frac{e^{2i\theta}}{i^k}+\frac{e^{-2i\theta}}{(-i)^k}\right),\quad\text{as }k\to\infty\,, \tag{12}$$

and so for $n=3$ the sum over k in Eq. (9) diverges. Consequently, this expression for w, with this ordering of sums, is divergent.

More importantly, Eq. (12) shows that no expansion of w in terms of $w_{n+1,k}$ can be absolutely convergent. Thus the best that one can hope for is to rearrange the terms in the sums in Eq. (9) to get conditional convergence. It is questionable, however, that such an answer, if it existed, would be meaningful, since any additional rearrangement would then produce a different answer [14]. In particular, for any coefficient multiplying $Ie^{im\theta}$, as m varies over even numbers, we would be able to get any number desired.

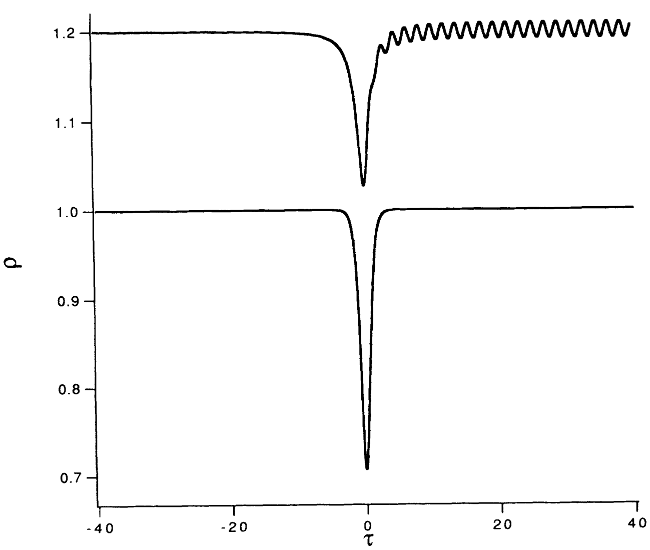

FIG. 1. $\rho(\tau)$ vs τ for a potential that generates reflections, $\omega^2(\tau)=(\tau^2+2)/(\tau^2+1)$ (top plot), and for a reflectionless potential, $\omega^2(\tau)=1+2\epsilon^2\text{sech}^2\tau$ (bottom plot). The values of ϵ are 0.6 (top) and 1 (bottom). For clarity the top curve displays $\rho(\tau)+0.2$.

Finally, we turn to an iterative procedure for solving Eq. (3) [5,7]. (To establish contact with Ref. [7], our ρ_n should be replaced by $\Omega^{(n)^{-1/2}}$.) The iteration formula is

$$\rho_{n+1}=[\omega^2+(1/\rho_n)\epsilon^2\ddot{\rho}_n]^{-1/4}\,. \tag{13}$$

Typically, $\rho_0=\omega^{-1/2}$. To see whether this procedure converges to the exact solution, ρ_{ex}, in the reflectionless case, we linearize Eq. (13) about ρ_{ex} and examine the linear stability of this fixed point. [For specificity, ρ_{ex} and ω can be taken to be the ones corresponding to the example of Eq. (7). The result obtained, however, is independent of this choice.] Writing $\rho_n=\rho_{\text{ex}}+\delta_n$, to first order in δ_n, Eq. (13) becomes $\delta_{n+1}=G\delta_n$, where G is the operator

$$\frac{1}{4}(1-\omega^2\rho_{\text{ex}}^4)-\frac{\epsilon^2}{4}\rho_{\text{ex}}^4\frac{d^2}{d\tau^2}\,. \tag{14}$$

To see the eigenvalue spectrum of G, we examine the eigenvalue equation $Gu_\lambda=\lambda u_\lambda$, with the boundary conditions that u is bounded at $\tau=\pm\infty$. Then, since ρ_{ex}^{-4} is positive, and $\omega^2-\rho_{ex}^{-4}$ is bounded from above on the interval $(-\infty,+\infty)$, this is a Sturm-Liouville problem where the eigenvectors form a complete set on the interval $\tau\in(-\infty,+\infty)$, and the eigenvalues are real and unbounded from above [11]. Consequently, in the space of functions of τ, ρ_{ex} is a linearly unstable fixed point. We note that this statement is independent of the asymptotic behavior of ρ_{ex} as $\tau\to+\infty$. Thus the procedure diverges regardless of whether the potential ω^2 is reflectionless or not, provided our starting point is not on the stable manifold (the set of functions of τ which can be expanded in terms of u_λ with only $|\lambda|<1$ contributing).

We have carried out explicitly the first five iterations of Eq. (13) for $\rho_{ex}(\tau)$ and $\omega(\tau)$ given by Eq. (7), with $\rho_0(\tau)=\omega^{-1/2}(\tau)$. At $\epsilon^2=0.53$, for example, $|\rho_n(0)-\rho_{ex}(0)|$ takes on the values 0.0264, 0.0131, 0.0105, 0.0108, and 0.0116. The asymptotic nature of the iterants is clearly visible, and the optimum truncation here occurs at $n=2$. We have performed similar calculations for values of ϵ^2 between 0.01 and 1, and observed the same general behavior for $\epsilon^2>0.19$ (for smaller values of ϵ more iterations are needed to see the turnaround). The algebraic operations on functions were performed by the symbolic manipulation routine MATHEMATICA.

In summary, we have given precise conditions for the construction of a large class of reflectionless potentials $\omega^2(\tau)$ which are analytic in a strip about the real τ axis. To one such potential, we have applied adiabatic perturbation theory and shown that the perturbation series is not absolutely convergent. Similarly, we have found that a WKB-type iteration procedure is divergent. Thus the asymptotic character of the expansion (or the iteration) is a mathematical artifact of the method used, not related either to itegrability or to the potential being reflectionless.

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[a]Permanent address: Department of Applied Physics, Columbia University, New York, NY 10027.

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