PHYSICAL REVIEW LETTERS

VOLUME 68 13 APRIL 1992 NUMBER 15

Instability Induced by Symmetry Reduction

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(Received 9 December 1991)

We demonstrate that instabilities in a Hamiltonian system can occur via deformations that reduce the symmetry of the system. The movement of eigenvalues at an equilibrium point of a family of Hamiltonian systems is constrained by the symmetry type of the system. If deformations of a family change the symmetry type, then instabilities can appear at multiple eigenvalues that produce large amplitude changes in the system dynamics. We illustrate this phenomenon in the context of a low-dimensional Hamiltonian normal form, and then analyze the instability of a vortex filament in a strain field.

PACS numbers: 03.20.+i

The concepts of stability in dissipative and conservative systems are quite different. In dissipative systems a steady state is stable if all eigenvalues of the linearized operator have negative real parts. In Hamiltonian systems the spectrum is symmetric with respect to the imaginary and real axes; therefore, a necessary condition for stability is that the spectrum lies entirely on the imaginary axis. Krein [1] described the movement of eigenvalues in generic, nonsymmetric Hamiltonian systems as free parameters in a system are varied. Simple eigenvalues remain on the imaginary axis under Hamiltonian perturbations but multiple eigenvalues lying on the imaginary axis split and leave the imaginary axis. The generic theory of Williamson [2] and Galin [3] proves that the passing of eigenvalues along the imaginary axis does not occur in generic one- and two-parameter families of Hamiltonian systems. However, the movement of eigenvalues in generic families of symmetric Hamiltonian systems is affected by their symmetry type (see Dellnitz, Melbourne, and Marsden [4]). For example, for the symmetry group S^1 acting on $\mathbf{R}^2$, the origin is always an equilibrium and its eigenvalues always lie on the imaginary axis. Therefore, multiple eigenvalues at the origin pass along the imaginary axis in this case (Golubitsky and Stewart [5]).

The next section describes a simple example we have found of instability induced by symmetry reduction when the symmetry group of a system is reduced from S^1 to Z_2. While the linear stability of the origin is affected only slightly by the deformation, there are large magnitude changes in other features of the system behavior. For the motion associated to a vortex filament of an ideal fluid in a strain field, multiple eigenvalues occur when varying an axial wave number. When the symmetry is reduced from S^1 to Z_2 by imposing a strain field, we show how our theory applies.

Steady Hamiltonian bifurcations in $\mathbf{R}^2$.—The group S^1 acts on the Euclidean plane (with complex coordinate A) by rotations. Vector fields that are symmetric with respect to this action can be written in the form $\dot{A}=f(|A^2|)A$ with f: $\mathbf{R}^+ \rightarrow \mathbf{C}$. This vector field is Hamiltonian if the values of f are pure imaginary. For such a Hamiltonian vector field, the flow is along concentric circles centered at the origin. The origin is always an equilibrium and its eigenvalues are on the imaginary axis. After a possible transformation of the parameter space, a generic family with S^1 symmetry can be written as $\dot{A}=i(\lambda A+a|A^2|A)+o(|\lambda A|,|A^3|)$ with $a,\lambda \in \mathbf{R}$. For $a\lambda<0$, there is a circle of equilibrium points that collapses onto the origin as $\lambda \rightarrow 0$. The flow on the remaining circles near the origin is a rotation with angular frequency that depends on radius.

We consider perturbations that reduce the symmetry group from S^1 to Z_2, with the action of Z_2 given by $A \to -A$ in the complex plane. Generic deformations of an S^1 symmetric family that reduce the symmetry group to Z_2 can be written in the form

$$\dot{A} = i(\lambda A + a|A^2|A) + \epsilon[b\bar{A} + c(A^3 - 3A\bar{A}^2) + d\bar{A}^3] + o(|\epsilon|, |\lambda A|, |A^3|)$$

with $\lambda, a, \epsilon \in \mathbf{R}$ and $b,c,d \in \mathbf{C}$. By scaling A, t, and ϵ, we can assume that $a=-1$ and $b=-i$ if a and b are nonzero, and that the system is a small perturbation of $\dot{A} = i(\lambda A - |A^2|A - \epsilon\bar{A})$.

There are three different types of phase portraits of $\dot{A} = i(\lambda A - |A^2|A - \epsilon\bar{A})$ encountered with varying λ and fixed $\epsilon > 0$. For $\lambda < -\epsilon$, there is a single equilibrium point at the origin, surrounded by a family of periodic orbits that fills the remainder of the plane. At $\lambda = -\epsilon$, the origin has a double zero eigenvalue and splitting of the eigenvalues occurs. In the range $-\epsilon < \lambda < \epsilon$, the origin is a saddle, and its separatrices form a figure eight. There are two stable equilibria on the $\mathrm{Im}(A)$ axis that bifurcated from the origin. At $\lambda = \epsilon$, another bifurcation occurs and the origin becomes stable again. A pair of saddle equilibria bifurcate from the origin along the $\mathrm{Re}(A)$ axis. The separatrices of these equilibria form a figure topologically equivalent to two circles intersecting at two points. These separatrices divide the plane into four regions that are neighborhoods of the three stable equilibria and infinity. In two regions, there are stable equilibria on the $\mathrm{Im}(A)$ axis at $\mathrm{Im}(A) = \pm\sqrt{\lambda + \epsilon}$. The family of phase portraits obtained from this one-parameter family is stable within the class of Z_2 symmetric Hamiltonian systems, so perturbation of the family by reincorporating neglected terms will not change the form of the bifurcations that occur for small ϵ. Figure 1 shows the three different types of phase portraits.

Comparing these phase portraits for the Z_2 symmetric normal form with the ones in the S^1 symmetric case, we see that there is a continuous evolution of a stable equilibrium of growing amplitude beginning with $\lambda = -\epsilon$. Fluid systems have some dissipation, so we investigate the addition of damping to the model. The model becomes $\dot{A} = i(\lambda A - |A^2|A - \epsilon\bar{A}) - \delta A$. If the damping coefficient δ is small compared to the coefficient ϵ of the symmetry-breaking deformation, the bifurcations of the model with varying λ are essentially unchanged. There are two pitchfork bifurcations at the origin. The stable equilibria of the Hamiltonian system become sinks and the saddle separatrices form the boundaries of the basins of attractions of the sinks. In the parameter region in which there are three sinks, numerical computations indicate that the basin of attractions of the nonzero sinks are substantially larger than those of the origin (see Fig. 2).

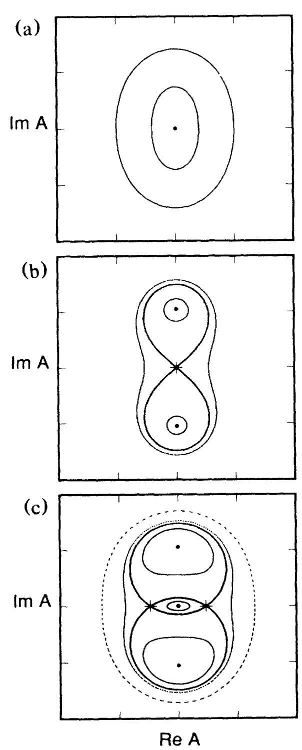

FIG. 1. Hamiltonian normal forms $\dot{A} = i(\lambda A - |A^2|A - \bar{A})$ with (a) $\lambda = -1.5$, (b) $\lambda = 0.1$, and (c) $\lambda = 1.5$.

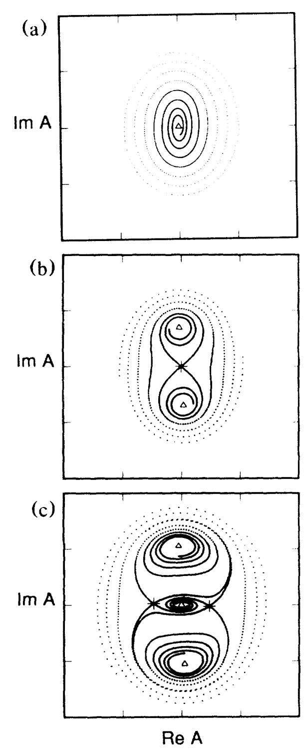

FIG. 2. Dissipative perturbations $\dot{A} = i(\lambda A - |A^2|A - \bar{A}) - 0.1A$ of the Hamiltonian normal forms with (a) $\lambda = -1.5$, (b) $\lambda = 0.1$, and (c) $\lambda = 1.5$.

Note that the addition of dissipation to S^1 symmetric systems make the origin a global attractor with Lyapunov function $|A|^2$.

Vortex instability induced by symmetry reduction. —We turn now to the physical problem of vortex motion with the action of weak irrotational plane strain. Linear stability analysis of this problem was considered by Moore and Saffman [6] and Tsai and Widnall [7]. In the absence of strain, a vortex of finite circular cross section is stable. It becomes unstable by the action of weak irrotational plane strain. Moore and Saffman [6] find that "from a mathematical point of view, we can associate the instability with the degeneracy caused by two physically distinguishable eigenmodes of the unperturbed vortex having the same eigenfrequency." Instability occurs for wavelength and frequencies at the intersection points of the dispersion curves of two distinct eigenmodes for an isolated vortex. Moreover, these modes are coupled with the geometric mode of deformation $\cos 2\phi$, $\sin 2\phi$. This means that their azimuthal wave numbers satisfy $|n_1 - n_2| = 2$. The case $n_1 = 1$, $n_2 = -1$ is of special interest, corresponding to bending waves in which fluid particles are deflected off the axis in a fixed plane. The bending waves have been observed in experiments (Fig. 3). We use our analysis of instability due to symmetry reduction to explain these observations.

The symmetry group of the undeformed vortex problem is $S^1 \times O(2)$. This reflects the fact that the problem is invariant under rotations in the azimuthal direction and under translations and reflections in the axial direction. We assume that the axial wave number k corresponds to an intersection point of the dispersion curves for an isolated vortex (see Tsai and Widnall [7]). There are two eigenfunctions corresponding to the double zero eigenvalue: $\exp[i(\phi + kz)]u_1(r)$ and $\exp[i(\phi - kz)]u_2(r)$. Let A and B be complex amplitudes corresponding to these modes and $\{\alpha,\beta\} \in S^1 \times O(2)$. Then $\{\alpha,\beta\}$ acts on the amplitudes (A,B) in the following way:

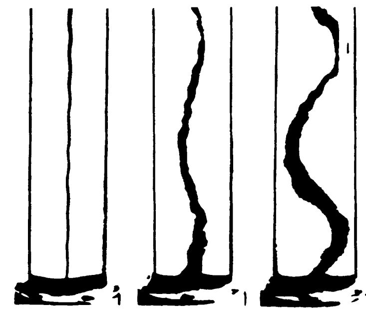

FIG. 3. Observations of a vortex filament in a strain field, from Vladimirov and Tarasov [8]. The experiments are described in the text.

$$\{\alpha,\beta\}(A,B) = (\exp[i(\alpha + k\beta)]A, \exp[i(\alpha - k\beta)]B). \quad (1)$$

Also, we are allowed to interchange amplitudes A and B:

$$\tau(A,B) = (B,A). \quad (2)$$

Amplitude equations that respect (1) and (2) have the form

$$\frac{dA}{dt} = s_0 A + A(s_1|A|^2 + s_2|B|^2) + \text{h.o.t.},$$
$$\frac{dB}{dt} = s_0 B + B(s_2|A|^2 + s_1|B|^2) + \text{h.o.t.}, \quad (3)$$

where h.o.t. in this equation and those below denotes higher-order terms. We assume that the symmetry is reduced to S^1 to Z_2 in the azimuthal direction by the action of weak irrotational plane strain. Then degenerate eigenvalues at the origin for which the eigenspaces are coupled with the geometric mode of deformation produce splitting. New terms appear in the amplitude equations that reflect the splitting. For the smaller symmetry group, the normal form is

$$\frac{dA}{dt} = s_0 A + \epsilon p_0 \bar{B} + A(s_1|A|^2 + s_2|B|^2) + \epsilon(p_1 A^2 B + p_2 \bar{A}\bar{B}^2 + p_3|A|^2\bar{B} + p_4|B|^2\bar{B}) + \text{h.o.t.},$$
$$\frac{dB}{dt} = s_0 B + \epsilon p_0 \bar{A} + B(s_2|A|^2 + s_1|B|^2) + \epsilon(p_1 B^2 A + p_2 \bar{A}^2\bar{B} + p_3|B|^2\bar{A} + p_4|A|^2\bar{A}) + \text{h.o.t.}, \quad (4)$$

with ϵ a measure of the departure from full $S^1 \times O(2)$ symmetry (e.g., the eccentricity of elliptical cross sections). We note that the additional cubic terms are small in comparison with the terms $\epsilon\bar{A}$, $\epsilon\bar{B}$, $A|A|^2$, $A|B|^2$, $B|A|^2$, and $B|B|^2$. Therefore, we pass to a limit in which Eqs. (4) become

$$\frac{dA}{dt} = s_0 A + \epsilon p_0 \bar{B} + A(s_1|A|^2 + s_2|B|^2) + \text{h.o.t.},$$
$$\frac{dB}{dt} = s_0 B + \epsilon p_0 \bar{A} + B(s_2|A|^2 + s_1|B|^2) + \text{h.o.t.} \quad (5)$$

Systems (4) and (5) have an invariant subspace $A = B$. On this invariant subspace, denote $C = A = B$. Then (5) becomes

$$\frac{dC}{dt} = s_0 C + \epsilon p_0 \bar{C} + (s_1 + s_2)C|C|^2 + \text{h.o.t.} \quad (6)$$

Equation (6) is the example treated above.

We relate this analysis to observations of Vladimirov and Tarasov [8]. They studied a draining-vortex flow in a cylindrical vessel with an elliptical cross section of small eccentricity. A number of tests were conducted to measure the instability of the vortex flow to bending motions. In a typical experiment, a vessel containing water, filled to the level $L+\Delta L$, was set into a state of rigid rotation with a velocity Ω. Then a hole located at the center of the bottom of the vessel was opened. A straight vortex formed in the center of the cylinder and was made visible by the introduction of dye. When the level fell to height L, the hole was covered and the rotation was stopped. After a short interval of time, the flow reorganized itself from a draining vortex to one with a zero axial velocity component. The vortex remained straight in certain ranges of L, while in others there was instability manifest in bending of its axis (Fig. 3). In all cases the bends in the vortex core were stationary in the laboratory reference frame. The ranges of the cylinder heights for which instability occurred were consistently longer than those predicted by the linear theory. This fact is explained by the Hamiltonian unfolding shown in Fig. 1 since the nontrivial steady states occur for $s_0 > -\epsilon p_0$.

Discussion.— Instability induced by symmetry reduction is a new, general phenomenon we have discovered in Hamiltonian and weakly dissipative systems. Normal form analysis that takes into account the effects of symmetry can explain how small symmetry reduction results in large magnitude changes within a dynamical system. We have applied such an analysis to account for instabilities of a vortex filament in a noncircular cylinder. Though we illustrated this effect with one simple example, there are many examples of instabilities of Hamiltonian systems induced by symmetry reduction. For example, if one allows axial flow in the vortex experiments described above, the bifurcation shifts to nonzero points along the imaginary axis and we have "Hamiltonian Hopf bifurcation." Splitting of eigenvalues due to symmetry reduction occurs at this Hamiltonian Hopf bifurcation when a strain field is imposed by deformation of the cylinder.

The research of J.G. was partially supported by the National Science Foundation and the Air Force Office of Scientific Research. The research of A.M. was partially supported by the National Science Foundation, DMS 89-19074 and CTS 89-06343. We would like to thank Phil Marcus, Jerry Marsden, and Vladimir Vladimirov for very helpful discussions.

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