Quasienergies, Stark Hamiltonians, and Growth of Energy for Driven Quantum Rings

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We study time-dependent Schrödinger operators in Aharonov-Bohm geometries where the flux threading the hole increases linearly with time. We show that the quasienergy operator has, in these cases, the same spectrum as the *time-independent* Stark Hamiltonian on the *universal covering space.* Combining known results on Stark Hamiltonians with a theorem of Bellissard, we prove that the energy of a particle on a *finite* ring, with smooth background potential, increases without bound as $t \to \infty$.

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Schrödinger operators that describe quantum particles acted on by external *time-independent* electric and magnetic fields may or may not be time independent. When the domain is simply connected, for example, for the Euclidean space, there is always a choice of gauge so that the Schrödinger operator is time independent. We shall call this the *Stark form* of the operator. When the domain is not simply connected, as, for example, in the case of Aharonov-Bohm geometry, there is no time-independent form for the operators describing (interacting or noninteracting) particles driven by a time-independent electromotive force due to a linearly increasing flux threading the hole [1].

The Hamiltonians of classical mechanics, in contrast, can always be brought into a time-independent form when the external fields are time independent. The price one has to pay for doing so is to replace the domain Ω with a hole by its (universal) covering space Ω^c, which is simply connected and which is where the static potential lives. Recall that the (universal) covering spaces of the circle and the annulus are the infinite line and the infinite helix, respectively.

The Schrödinger operators associated with time-independent electric and magnetic fields on the (universal) covering space can always be brought into a Stark form. However (unlike the case in classical mechanics), it is not *a priori* clear what these operators have to do with the original problem with the Aharonov-Bohm geometry. Our purpose here is to show that the Stark operator for the (universal) covering space is the quasienergy operator for the Aharonov-Bohm geometry. (This will be explained in some detail below.) We shall then apply this observation to give a proof which is both elementary and rigorous of the fact that the energy of a particle on a driven thin ring increases without bound for a large class of background potentials and *all* initial conditions.

For the sake of simplicity, let us focus on the one-particle Schrödinger operator without magnetic fields on Ω [2]:

$$H(t)=\frac{1}{2m}[-i\hbar\nabla - t\mathbf{a}(\mathbf{x})]^2+V(\mathbf{x}),\qquad \nabla\times\mathbf{a}=0,\quad \mathbf{x}\in\Omega. \tag{1}$$

The electric field, $e\mathbf{E}(\mathbf{x})=\mathbf{a}(\mathbf{x})-\nabla V(\mathbf{x})$, and magnetic field (which vanishes identically on Ω) are both time independent.

The magnetic flux threading the hole is linearly increasing with time and we choose the unit of time so that after $t=1$ the flux through the hole increases by the unit of quantum flux. This sets the normalization of the loop integral of the vector potential: $\oint \mathbf{a}(\mathbf{x})\cdot d\mathbf{x}=2\pi\hbar$.

Since $\nabla\times\mathbf{a}=0$, $\mathbf{a}$ is locally the gradient of a function

$$\Lambda(\mathbf{x}_c)\equiv\int^{\mathbf{x}_c}\mathbf{a}(\mathbf{x}')\cdot d\mathbf{x}'. \tag{2}$$

$\Lambda(\mathbf{x}_c)$ is (in general) a function of the (universal) covering space Ω^c, where the Hamiltonian $H(t)$ is related by a gauge transformation to a time-independent Stark Hamiltonian. Indeed, let the gauge transformation be

$$G(t)\equiv\exp\{i[t\Lambda(\mathbf{x}_c)/\hbar]\}, \tag{3}$$

where $\mathbf{x}_c$ denotes a point in Ω^c. In the new gauge the Hamiltonian has the Stark form [3]:

$$\begin{aligned}H_{\text{Stark}}&\equiv G^\dagger(t)H(t)G(t)-i\hbar G^\dagger(t)[\partial_t G(t)]\\&=-\frac{\hbar^2}{2m}\Delta+V(\mathbf{x}_c)+\Lambda(\mathbf{x}_c).\end{aligned} \tag{4}$$

$V(\mathbf{x}_c)+\Lambda(\mathbf{x}_c)$ is a periodic plus linear (equivalent to the washboard) potential, on the covering space. Because of the presence of a periodic potential, we dub such operators Wannier-Stark Hamiltonians [4].

The gauge transformation $G(t)$ is well defined on Ω only for *integer* times and cannot be substituted in the time-dependent Schrödinger equation, as is evident from Eq. (4), where differentiation with respect to time is taken. It is then natural to ask what is the relation, if any, between the time-independent Stark Hamiltonian H_{Stark} on the covering space and the time-dependent one on the ring.

The evolution generated by *general* time-dependent Hamiltonians (including cases where $\mathbf{E}$ and $\mathbf{B}$ *are* time dependent) can be reduced to considering time-independent ones. In classical mechanics the price is that the time-independent Hamiltonians are defined on a larger phase space, where E and t are the additional conjugate coordinates (see, e.g., [5]). In quantum mechanics, the analogous procedure, due to Howland [6], is to enlarge the Hilbert space to $L^2(\Omega\otimes R)$, with elements $\psi(\mathbf{x},s)$, $\mathbf{x}\in\Omega$, and $s\in R$. On this larger (grand) Hilbert space, with one extra coordinate, one considers the (quasienergy) operator

$$\mathbf{K}\equiv -i\hbar\partial_s + H(s) . \tag{5}$$

It has the property

$$\exp[-i(\sigma\mathbf{K}/\hbar)\psi](\mathbf{x},s) = U(s,s-\sigma)\psi(\mathbf{x},s-\sigma) , \tag{6}$$

where $U(t,s)$ is the unitary propagation operator from time s to time t for $H(t)$.

In the case of Eq. (1), $H(t)$ has the property of being periodic *up to unitary* [7], i.e.,

$$H(t+1) = G^{\dagger}H(t)G , \tag{7}$$

where $G\equiv G(1)$. This property is inherited by $\mathbf{K}$. The *analog* of the usual Bloch-type analysis then says that the spectral analysis of $\mathbf{K}$ reduces to the study of the spectra of $\mathbf{K}$ restricted to the spaces of "Bloch waves" in the s variable. In particular, the analog of the periodic Bloch waves, normalized as usual in $L^2(\Omega\otimes[0,1])$, are those that satisfy the condition [8]

$$\psi(\mathbf{x},s+1) = G^{\dagger}\psi(\mathbf{x},s) . \tag{8}$$

Combined with Eq. (6) this gives

$$\begin{aligned}\exp[-i(\mathbf{K}/\hbar)\psi](\mathbf{x},s) &= U(s,s-1)G\psi(\mathbf{x},s)\\ &= GU(s+1,s)\psi(\mathbf{x},s) .\end{aligned} \tag{9}$$

We used $U(t+1,s+1) = G^{\dagger}U(t,s)G$, which is a direct consequence of Eq. (7). It follows from Eq. (9) that the evolution $U(s+1,s)$ can be studied via the evolution generated by $\mathbf{K}$ [9].

We shall now show that $\mathbf{K}$ is unitarily equivalent to the Wannier-Stark Hamiltonian H_{Stark} of Eq. (4), defined on the covering space $L^2(\Omega^c)$.

The Zak transform [10] from the Bloch states in the grand Hilbert space to the Hilbert space associated with the covering space Ω^c is

$$\tilde{\psi}(\mathbf{x}_c) \equiv \int_0^1 ds\, \psi(\mathbf{x},s)\exp\{i[s\Lambda(\mathbf{x}_c)/\hbar]\} . \tag{10}$$

$\mathbf{x}_c\in\Omega^c$ are the preimages of $\mathbf{x}\in\Omega$. The inverse transform is

$$\psi(\mathbf{x},s) = \frac{1}{2\pi\hbar}\sum_{\text{preimages of } \mathbf{x}} \tilde{\psi}(\mathbf{x}_c)\exp\{-i[s\Lambda(\mathbf{x}_c)/\hbar]\} . \tag{11}$$

Equations (10) and (11) are compatible with the boundary conditions, Eq. (8), and preserve the appropriate norms.

It is a simple exercise, using Eqs. (10) and (11), and the boundary condition, Eq. (8), to show that for the operators appearing in Eqs. (1) and (5) one has

$$\begin{aligned}&\widetilde{(V\psi)}(\mathbf{x}_c) = V(\mathbf{x}_c)\tilde{\psi}(x_c) ,\\ &[(-i\hbar\tilde{\nabla}-s\mathbf{a})\psi]^{\sim}(\mathbf{x}_c) = (-i\hbar\nabla_{x_c}\tilde{\psi})(\mathbf{x}_c) ,\\ &(\widetilde{-i\hbar\partial_s\psi})(\mathbf{x}_c) = \Lambda(\mathbf{x}_c)\tilde{\psi}(x_c) .\end{aligned} \tag{12}$$

It follows that $\mathbf{K} = H_{\text{Stark}}$, i.e., the Stark Hamiltonian is the quasienergy operator. This is our main result and it extends to interacting electrons as well.

There is a basic intuition from tunneling that says that Stark operators do not have normalizable eigenvectors. This intuition has been established rigorously for a wide class of potentials [4,5,11]. In particular, it is known that in one dimension, if, e.g., V is twice differentiable, the spectrum of Wannier-Stark Hamiltonians is absolutely continuous—there are no normalizable eigenstates [12].

The question whether the energy of a particle in a driven ring is bounded or not bears on the question whether idealized rings (i.e., without inelastic processes) provide a model for dissipation. Partly because of this it had been studied by many authors, using various techniques, including simulations and approximate and analytic methods. Some of these authors arrived at conflicting conclusions [13–15]. Recently, Gefen and Thouless [14] studied the growth of energy in driven rings by considering Zener tunneling between the energy bands of the (adiabatic) spectra of $H(t)$. They have shown that provided the gaps are random and uncorrelated, a phenomenon related to Anderson localization takes place in energy space which keeps the energy of the particle bounded.

We shall prove that for a one-dimensional ring with any smooth (twice differentiable will do) background potential the energy will, eventually, run away. That this is so can also be seen from localization theory if one takes into account the asymptotic decrease of the gaps [15,16]. The ultimate growth of the energy is therefore something that can be seen in more than one way. The point we want to make is partly that of simplicity and rigor, partly to settle an issue that had been somewhat controversial, and mostly to illustrate the use of the relation between the time-dependent Hamiltonian on the ring and the Stark Hamiltonian on the line. The strategy is closely related to the one in [15] where Wannier-Stark Hamiltonians are studied. The basic tool is due to Bellissard [17] originally devised for time-periodic Hamiltonians. It makes use of two facts, that the Floquet operator M has no (normalizable) eigenvalues, and that $H(t=0)$ is bounded below and has *only* discrete spectrum whose point of accumulation is at infinity.

Let $|e_j\rangle$, $j=1,\ldots,\infty$, be the normalized eigenvectors of $H(t=0)$ with (ordered) eigenvalues E_j. Let $c(k,n)\equiv|\langle e_k|M^n|\psi\rangle|^2$, with $|\psi\rangle$ an initial normalized state of finite energy, $M\equiv GU(0,1)$. By the completeness of $|e_j\rangle$ and the unitarity of M,

$$\sum_{k=0}^{\infty} c(k,n) = 1 \tag{13}$$

for all integer n's. On the other hand,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}c(k,n)=|\langle e_k|P_{\rm pp}|\psi\rangle|^2=0\,, \tag{14}$$

where $P_{\rm pp}$ is a projection on the pure-point part of the spectrum of M. The first equality is the Wiener (RAGE) theorem (see [5]). The second equality is the statement that M has no pure-point part. Now use the fact that $U(n,0)=(G^\dagger)^n M^n$ together with Eqs. (7), (13), and (14) to get

$$\frac{1}{N}\sum_{n=0}^{N}\langle\psi(n)|H(n)|\psi(n)\rangle=\frac{1}{N}\sum_{n=0}^{N}\langle\psi|(M^\dagger)^nH(0)M^n|\psi\rangle=\frac{1}{N}\sum_{n=0}^{N}\sum_{k=0}^{\infty}E_kc(k,n)>\epsilon E_0+(1-\epsilon)E_j\,, \tag{15}$$

where $\epsilon\equiv(1/N)\sum_{n=0}^{N}\sum_{k=0}^{j-1}c(k,n)$ can be chosen as small as one wants, and j and thus E_j as large as one wants, provided N is large enough. This completes the proof that the energy is unbounded.

We close this paper with a sequence of remarks, mostly about open problems.

(1) The proof does not provide information on the rate of growth of energy: The Wiener theorem is "soft" and gives no *a priori* information on the rate of convergence to zero.

(2) It is instructive that the cases with infinitely large domains remain open. The proof fails because $H(t=0)$ may have essential spectrum in which case E_j can get stuck at finite energies even as $j\to\infty$. From a naive physical intuition it appears surprising that the energy growth can be hindered when the (adiabatic) spectrum becomes more dense (and ultimately is essential). An example where something related happens is a one-dimensional ring or radius R, with "random potential" V. From the Landauer formula for conductance one sees that (since the driving potential is independent of R) energy grows at a rate which *decreases* with R.

(3) Ao [15] gave a fascinating argument suggesting that the Dirac comb (Kronig-Penney model) is critical; that is, the spectrum is made of localized states for weak electric fields, and has no localized states if the field is large [18]. This would mean that for rings with the Dirac comb potential the energy will remain bounded (in fact, it will be an almost periodic function of time) for weak driving.

(4) By adapting standard methods from the scattering theory of Stark Hamiltonians [5,11] to the multidimensional, and also multiparticle case, the spectrum would have an absolutely continuous component. This would mean that for lots of finite-energy initial states, the energy would grow. In fact, the basic intuition that "reasonable" Stark Hamiltonians have no eigenvalues suggests that this will hold for *all* initial states [19].

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[1] For the sake of simplicity, we assume throughout that there is a single hole.

[2] We impose Dirichlet boundary conditions on the boundary $\partial\Omega$.

[3] When Ω is simply connected Ω and Ω^c coincide. Equation (4) gives for $\mathbf{a}=e\mathbf{E}$, the textbook form of a dc Stark effect $\mathbf{A}(\mathbf{x}_c)=e\mathbf{E}\cdot\mathbf{x}_c$.

[4] There is tremendous literature on Wannier-Stark Hamiltonians. The experimental situation is reviewed in G. Bastard, J. A. Brum, and R. Ferreira, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic, New York, 1991), Vol. 44. Rigorous mathematical results can be traced from G. Nenciu, Rev. Mod. Phys. **63**, 91 (1991); V. S. Buslaev and L. A. Dmitrieva, Leningrad Math J. **1**, 287 (1990); F. Bentosela *et al.*, J. Phys. A **21**, 3321 (1988); J. Howland and I. W. Herbst, Commun. Math. Phys. **80**, 23 (1981).

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