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Geometrical Phase and Amplitude Accumulations in Dissipative Systems with Cyclic Attractors

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We show that the geometrical (Berry) phases discovered in Hamiltonian systems can also be defined as resulting from parallel transportation of vectors for nonlinear dissipative systems with cyclic attractors. If the nonlinear dissipative systems possess a certain kind of asymptotic solution defined in this Letter, the phase and amplitude accumulation of a geometrical type can be defined. Detuned one- and two-photon lasers showing periodic intensity pulsations are taken as examples of such systems.

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Since the remarkable discovery by Berry [1] of the quite general existence of an observable phase accumulation in the wave function of a quantum-mechanical system with an adiabatically changing Hamiltonian, the understanding of this phase has gained greatly in both its deepness and its extensiveness. The connection of this phase with the parallel transportation of vectors along a curve was first realized by Simon [2], who supplied mathematical structure for the later generalization to various cases. The restriction to adiabaticity was lifted by Aharonov and Anandan [3] by removing the time integral of the expectation of the Hamiltonian as a dynamical phase from the wave functions. The evolution described by the resulting wave functions then defines a natural connection for any cyclic evolution of a quantum system. This idea was recently used by Samuel and Bhandari [4] to get rid of the restriction to the Hamiltonian system. These authors identified the real part of the expectation value of the linear operator as the dynamical frequency and established a natural connection for the linear nonunitary evolution. The restriction to cyclic motion was also removed by these authors [4]. Some later authors [5] treated the linear non-Hermitian systems by a more elegant approach using the technique of the biorthogonal set of state vectors. A "complex phase" was introduced by these authors. The phase has also been extensively studied by Chiao and co-workers [6] experimentally as well as theoretically. By assuming the existence of a global gauge invariance and charge conservation, Garrison and Chiao [6(b)] extended this phase concept for the multicomponent gauge-field theory with nonlinear equations derived from a Lagrangian. A subsequent Comment of Anandan pointed out that the gauge invariance is not necessary [7] for a one-component system with an overall amplitude accumulation. The case of multicomponent systems with gauge invariance was also considered there [7] to define a nonadiabatic and non-Abelian phase for nonlinear systems. In this Letter we show that neither unitariness of the evolution nor linearity of the systems is necessary for the existence of this geometrical phase in a multicomponent system. The concept of the geometrical phase can be extended to a quite different context, namely, dissipative nonlinear systems, where attractors exist. The essential requirement is that the considered system possesses a kind of *cyclic* solution for $t \rightarrow \infty$. If such asymptotic solutions (*cyclic attractor*) exist, the geometrical phase can be defined for a class of *dissipative* dynamical systems. The evolution of the system along such attractors will define a parallel transportation of vectors in some space. At the same time the concept of the geometrical amplitude accumulation is naturally introduced for a nonlinear system since we are dealing with dissipative systems where no conservation law guarantees the invariance of the amplitude during the parallel transportation of the vectors. This generalizes the complex phase defined in [5]. Detuned one- and two-photon lasers displaying periodic intensity pulsations will be taken as examples from laser physics.

Consider a dynamical system described by the following ordinary differential equations:

$$|\dot{\Psi}\rangle = |\mathcal{F}(|\Psi\rangle)\rangle\,, \tag{1}$$

where $|\Psi\rangle = (\psi_1, \psi_2, \ldots, \psi_n)$ is the vector describing the states of the system in phase space and $|\mathcal{F}(|\Psi\rangle)\rangle$ is a nonlinear vector-valued function $|\Psi\rangle$, which defines the evolution in the phase space. In general, $|\mathcal{F}(|\Psi\rangle)\rangle$ is also a function of the externally controllable parameters, e.g., the pumping in lasers.

We define now a diagonal matrix, $\mathcal{T}(\phi(t))$, whose elements are given by $(\exp[i\alpha_1\phi(t)], \exp[i\alpha_2\phi(t)], \ldots, \exp[i\alpha_n\phi(t)])$, where α_i's are real numbers and $\phi(t)$ is a real function. Suppose that for certain sets of parameters there exists a kind of asymptotically stable solution of Eq. (1), such that, for certain initial conditions and for $t \to \infty$, the following relation holds:

$$|\Psi(t+T)\rangle = \mathcal{T}(-\delta\phi)|\Psi(t)\rangle\,, \tag{2}$$

where $\delta\phi$ is a real quantity [8]. In the following our discussion will be based on this kind of solution and the evolution of the system (1) will correspondingly be restricted to a certain set of parameters, for which the solutions of type (2) exist. Such motions described by $|\Psi\rangle$ satisfying (2) are defined as *cyclic* attractors in this Letter. Obviously this is a generalization of the *cyclic* motion used by the earlier authors [3,4], where $\mathcal{T}$ is unity. Correspondingly, we define a periodic motion (limit-cycle solution) through

$$|\bar{\Psi}(t)\rangle = \mathcal{T}(\phi(t))|\Psi(t)\rangle\,. \tag{3}$$

In order to ensure the periodicity $|\bar{\Psi}(t+T)\rangle = |\bar{\Psi}(t)\rangle$ we require the relation

$$\mathcal{T}(\phi(t+T) - \phi(t) - \delta\phi) = \mathcal{T}(0) = I \tag{4}$$

or the relation

$$\phi(t+T) - \phi(t) = \delta\phi\,, \tag{5}$$

where I is the identity matrix. Denoting the space formed by all cyclic vectors $|\Psi\rangle$ as space $\mathcal{N}$ (total space), and that formed by periodic vector $|\bar{\Psi}\rangle$ as $\mathcal{R}$ (base space), we can also consider the relation (3) as a mapping (denoted as $\Pi: |\Psi\rangle \to |\bar{\Psi}\rangle$) from the space $\mathcal{N}$ to the so-called *ray* space $\mathcal{R}$. Obviously a cyclic motion draws a closed *curve* in the ray space $\mathcal{R}$, whereas such a closed curve in $\mathcal{R}$ generally corresponds to an open curve in $\mathcal{N}$. Roughly speaking, this is the origin of the phenomenon of anholonomy.

To proceed further we define a diagonal matrix Λ, whose elements are given by $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. We form a new vector

$$|\Phi\rangle = \exp\left[\Lambda \int^t \chi_d(t')dt'\right]|\Psi\rangle\,, \tag{6}$$

where a *dynamical* contribution χ_d is removed from the original vector and is defined by

$$\chi_d(t) = -\frac{\langle\Psi|\mathcal{F}(|\Psi\rangle)\rangle}{\langle\Psi|\Lambda|\Psi\rangle} \equiv \gamma_d + i\omega_d\,, \tag{7}$$

which is a complex quantity with its imaginary part defining a dynamical frequency and its real part a dynamical amplitude. Obviously the new vector is no longer cyclic in the sense of (2), because $|\Phi(t)\rangle$ accumulates itself both in amplitude and in phase. From (2), (6), and (7) we obtain

$$|\Phi(t+T)\rangle = \mathcal{T}\left(-\delta\phi - i\int_t^{t+T} \chi_d(\tau)d\tau\right)|\Phi(t)\rangle\,, \tag{8}$$

which defines total phase and amplitude accumulations in a period T.

A corresponding periodical vector can be defined through

$$|\bar{\Phi}(t)\rangle = \mathcal{T}(\bar{\phi}(t))|\Phi(t)\rangle\,, \tag{9}$$

where

$$\bar{\phi}(t) = \int^t [\nu(t') + i\gamma_d(t')]dt'\,. \tag{10}$$

The periodicity of $|\bar{\Phi}(t)\rangle$ requires

$$\int_t^{t+T} \nu(t')dt' = \delta\phi - \int_t^{t+T} \omega_d(\tau)d\tau\,. \tag{11}$$

Now our total space $\mathcal{N}'$ is spanned by $|\Phi\rangle$ and the base space $\mathcal{R}'$ by $|\bar{\Phi}\rangle$. The structure group is $\mathcal{T}(\bar{\phi}(t))$, which does not form a unitary group. A fiber consists of all vectors of the space $\mathcal{N}'$ which correspond to a vector in the base space through (9). The evolution of the system over one period draws a closed curve in the base space $\mathcal{R}'$, whereas the lifting of the such a closed curve up to the total space is generally open. In our present case it has both phase and amplitude anholonomy. To show that this lifting is horizontal we consider the evolution in these new spaces $\mathcal{N}'$ and $\mathcal{R}'$.

Using (1) we can derive the following evolution equations:

$$|\dot{\Phi}\rangle = \chi_d(t)\Lambda|\Phi\rangle + \exp\left[\Lambda\int^t \chi_d(t')dt'\right]|\mathcal{F}(|\Psi\rangle)\rangle\,, \tag{12}$$

$$|\dot{\bar{\Phi}}\rangle = i\dot{\bar{\phi}}(t)\Lambda|\bar{\Phi}\rangle + \mathcal{T}(\bar{\phi}(t))|\dot{\Phi}\rangle\,. \tag{13}$$

The contraction of the first equation with $\langle\Phi|\mathcal{T}^+(\bar{\phi}) \times \mathcal{T}(\bar{\phi})$ under consideration of (7) leads to

$$\langle\Phi|\mathcal{T}^+(\bar{\phi})\mathcal{T}(\bar{\phi})|\dot{\Phi}\rangle = 0\,, \tag{14}$$

whereas the contraction of the second equation with $\langle\bar{\Phi}|$ leads to

$$-i\frac{\langle\bar{\Phi}|\dot{\bar{\Phi}}\rangle}{\langle\bar{\Phi}|\Lambda|\bar{\Phi}\rangle} = \dot{\bar{\phi}}\,. \tag{15}$$

After integration of (15) over one period we have

$$\int_t^{t+T} \gamma_d = -\int_t^{t+T} d\tau\, \mathrm{Re}\left\{\frac{\langle\bar{\Phi}|\dot{\bar{\Phi}}\rangle}{\langle\bar{\Phi}|\Lambda|\bar{\Phi}\rangle}\right\}, \tag{16}$$

$$\delta\phi = \delta\phi_g + \delta\phi_d\,, \tag{17}$$

where $\delta\phi_g$ is the geometrical phase given by

$$\delta\phi_g = \int_t^{t+T} d\tau\, \mathrm{Im}\left\{ \frac{\langle \bar{\Phi} | \dot{\bar{\Phi}} \rangle}{\langle \bar{\Phi} | \Lambda | \bar{\Phi} \rangle} \right\}. \qquad (18)$$

$\delta\phi_d$ is the integral of the dynamical frequency over one period:

$$\delta\phi_d = \int_t^{t+T} \omega_d\, d\tau\,. \qquad (19)$$

Formulas (14) and (15) constitute the main result of this Letter. It is useful to interpret the above results using the language of differential geometry. Equation (13) represents a decomposition of a tangent vector into a vertical (first term) and a horizontal (second term) part, while Eq. (14) defines a connection, i.e., a horizontal space being orthogonal to the vertical subspace. Furthermore, the path integral of (15) along the closed curve in $\mathcal{R}'$ is a geometrical quantity, whose imaginary part given by (18) is the geometrical phase. The real part given by (16) corresponds to a geometrical amplitude accumulation. As stated by (16), this geometrical amplitude accumulation has the same magnitude as but opposite sign to the dynamic part, as required by the cyclic relation (2). From (17) we see that the whole phase accumulation is given by the summation of a dynamical and a geometrical part. For any cyclic evolution in the sense of (2) we can define a $|\Phi\rangle$ through (6) such that the actual evolution of the system defines a parallel transportation of vectors in the space $\mathcal{N}'$. The projection of this evolution onto the space $\mathcal{R}'$ will then generate a closed curve. The line integral along this closed curve gives the geometrical phase. The geometrical properties of these formulas have been discussed by many authors cited above. Our aim is then to give some interesting application of the above results to nonlinear dissipative systems with cyclic attractors.

As a first application we take the detuned single-photon laser with running wave configuration and homogeneously broadened atomic lines, which was shown [9] to be identical to the complex Lorenz equations. Using the notations of [9] and denoting the electric-field amplitude by X, a mixture of the field amplitude and the polarization by Y, and the inversion by Z, the set of equations is given by

$$\dot{X} = -kX + kY\,, \qquad (20)$$

$$\dot{Y} = -aY + (r - Z)X\,, \qquad (21)$$

$$\dot{Z} = -bZ + \tfrac{1}{2}(X^*Y + XY^*)\,, \qquad (22)$$

where $r = r_1 + ir_2$, $a = 1 + ir_2$. r_1 is related to the pumping. $r_2 = (1-k)\Delta$ and Δ is the detuning between the cavity and the atomic frequencies. b and k are the relaxation constants of the cavity and of the population inversion scaled by the relaxation constant of the polarization, respectively. The reference frequency is the cw frequency so that any new frequency will be due to the pulsations of the intensity. Here $|\Psi\rangle$ is a vector with three components (X,Y,Z) and the α_i are given by ($\alpha_1=1$, $\alpha_2=1$, $\alpha_3=0$). From our earlier analysis [10] we know that this vector is cyclic after the second threshold, when the intensity shows periodic pulsations. Using formula (7) the dynamical frequency is given by

$$\omega_d = -\frac{-r_2|Y|^2 + \mathrm{Im}[(r - Z - k)XY^*]}{|X|^2 + |Y|^2}\,. \qquad (23)$$

From this expression we see that $\omega_d = 0$ if there is no detuning, because Y is then asymptotically real and $r_2 = 0$. We know that there is no phase accumulation for perfectly tuned lasers in the domain of pulsation except for that induced by the cw frequency. Therefore there is no geometrical phase for the perfectly tuned lasers. For the case of detuning, the whole phase is given by [9]

$$\phi(t) = -k\int^t dt \frac{\mathrm{Im}[Y\exp(i\phi)]}{x_1}\,, \qquad (24)$$

where $x_1 = X\exp(i\phi)$ is a real variable. The dynamical phase is given by the integration of (23). The subtraction of the dynamical part from the total phase gives the geometrical phase. An example of these phases for the detuned case is shown in Fig. 1.

Our second example is the detuned two-photon laser. Denoting the electric-field amplitude by E, the polarization of the medium by P, and inversion by D, we can write the equations [11]

$$\dot{E} = (id - 1)kE - 2iPE^*\,, \qquad (25)$$

$$\dot{P} = -(id + 1)P + iDE^2\,, \qquad (26)$$

$$\dot{D} = b(D_0 - D) + 2i[PE^{*2} - P^*E^2]\,, \qquad (27)$$

where d is the scaled detuning parameter between the cavity and atomic frequencies and D_0 denotes the pumping rate. k and r have the same meaning as in the one-photon laser. Again we take the cw frequency as the reference frequency. It can be easily verified that α_i's are now given by ($\alpha_1=1$, $\alpha_2=2$, $\alpha_3=0$). The Hopf bifurca-

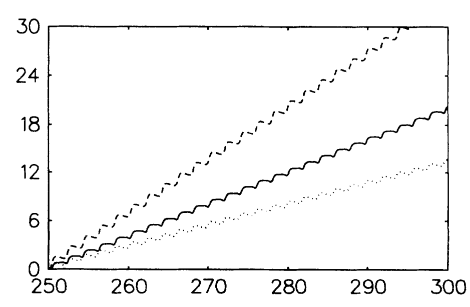

FIG. 1. The total phase (dashed line), dynamical phase (dotted line), and geometrical phase (solid line) with respect to the time τ scaled by the relaxation constant for the polarization for a detuned one-photon laser, where $k=4.0$, $b=0.1$, $\Delta=0.5$, and $r_1=91.0$.

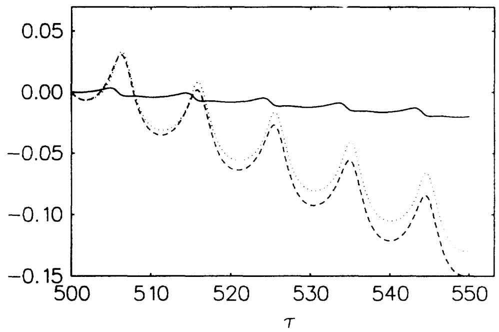

FIG. 2. The phases for a detuned two-photon laser with $k=0.3$, $b=0.05$, $d=0.1$, and $R=(D_0-D^{(0)})/D^{(0)}=18.0$, where $D^{(0)}$ is the stationary inversion. The meanings of the different curves are the same as in Fig. 1.

tion of this set of equations was studied in [11] and there exists a threshold for self-pulsing of the intensity. The cyclicity of E and P as defined in (2) can be easily verified in the parameter region of the periodic intensity pulsations. The dynamical frequency can be calculated from Eq. (7):

$$\omega_d=\frac{d[k|E|^2-|P|^2]+\mathrm{Im}\{iDE^2P^*-2iPE^{*2}\}}{|E|^2+2|P|^2}. \qquad (28)$$

Again we see $\omega_d=0$ if $d=0$, because the polarization is asymptotically imaginary in this case and the second term in the numerator then vanishes. The geometrical phase is therefore zero in this case as well. In the case of detuning we can calculate the geometrical phase in a way similar to that in the one-photon laser. The total phase is now given by [11]

$$\phi(t)=\int^t d\tau[2\,\mathrm{Re}\{P\exp(2i\phi)\}-kd]. \qquad (29)$$

The total, geometrical, and dynamical phases are shown in Fig. 2.

From the applications of formula (18) to the above examples the relationship between the geometrical phase and the whole phase accumulation becomes clearer. In the earlier work [10] we expected an analogy between the two and made some comparative study. Here we know precisely that the whole phase also contains a dynamical part which could not be anticipated by the earlier method. But the fact still holds that the geometrical phase is closely connected with detuning.

Finally we would like to mention the recent work of Kepler and Kagan [12]. These authors studied the phase in dissipative systems with explicit time-dependent parameters and required that these parameters change adiabatically. We show in this Letter that autonomous dissipative systems can generate a cyclic attractor on its own. After the subtraction of a dynamical-like contribution the evolution of the systems defines a parallel transportation law along this attractor and results in an intrinsic geometrical phase. Neither the modulated external parameters nor the adiabaticity is necessary to define this phase for dissipative systems.

To conclude, we have shown that the geometrical formalism for the parallel transportation of vectors (14) and the geometrical phase expressed as a line integral (18) hitherto formulated for Hamiltonian systems or linear non-Hamiltonian systems can be suitably adapted to dissipative systems showing cyclic attractors. At the same time the concept of the geometrical amplitude accumulation is naturally and generally introduced for nonlinear dissipative systems, which generalizes the concept of the complex phase introduced in [5]. Although the dynamical and the geometrical amplitude accumulations exactly cancel each other in our case, it would be interesting to investigate the property of the geometrical part. This Letter gives, on the one hand, a more general extension of the existence of the geometrical phases and, on the other, an important application to laser physics. Since it has been claimed for a long time that the Berry phase is purely geometric, it will be very interesting to derive this phase from a geometrical point of view for nonlinear dissipative systems also. Furthermore, we believe that the study of the geometrical structure of the evolution of the dissipative dynamical systems is an interesting field in its own right, where much work still needs to be done. We hope that this work will stimulate more interest in applying such an approach to dynamical systems.

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[8] In fact, the case with complex $\delta\phi$ can also be similarly considered. In that case the dynamic and the geometrical amplitude accumulations will not cancel so that we have a net amplitude accumulation. Because no such physical examples are known to us, we do not consider this case here.

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