Avalanche Dynamics from Anomalous Diffusion

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(Received 5 June 1991)

Bak, Tang, and Wiesenfeld introduced a sandpile model to study the so-called self-organized critical phenomena [P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988)]. There were several proposals to connect this discrete cellular automaton model with diffusion processes. We show how one may interpret the dynamics of this model as a discretized version of an anomalous diffusion equation, and we study the time evolution of the model off criticality.

PACS numbers: 68.35.Fx, 05.60.+w, 05.70.Ln, 64.60.Ht

Recently, there has been a great interest in various models that display self-organized criticality (SOC) [1]. The term "criticality" refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena. "Self-organized" refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e., the critical state is an attractor of the dynamics. Bak, Tang, and Wiesenfeld [1] (BTW) suggested that there may be an intimate connection between scale invariance in the spatial and temporal domains, i.e., between the fractal shapes in nature [2] and the $1/f^{\alpha}$ noise [3].

As an illustration, BTW introduced a sandpile model [1] based on a cellular automaton algorithm. Avalanches generated by an external perturbation can be observed on all length and time scales, and the characteristic distributions obey power-law behavior. Large-scale computer simulations [4] gave the values of the exponents. Dhar [5] obtained important results for the so-called Abelian cell automaton models.

In the BTW model, the SOC behavior is a feature of the dynamics of the relaxation processes initiated by the external perturbations. We show that this dynamics is a discretized version of a specific anomalous diffusion equation, whose diffusion coefficient has a pronounced peak as a function of the state variable. Our guess is that this last feature is the main ingredient responsible for the genesis of the SOC state.

Previously, Tang and Bak [6] used a diffusion picture to argue that in the stationary state a single site excitation must be followed in average by $\sim L^2$ toppling events in an $L\times L$ two-dimensional lattice. Manna [4] measured the average avalanche size in the SOC state, and his results are in very good agreement with the above prediction. Zhang [7] wrote down a relaxation equation for the correlation function, and gave exact solutions in $d\leq 4$ dimensions. In an attempt to describe the anomalously large fluctuations in the BTW and similar systems, others [8,9] studied correlations in systems described by diffusion equations with different nonlinear corrections and driven by white noise. However, a direct connection between any self-organizing model and these simple driven diffusions has not been made. A still open question is whether these hydrodynamical limits are in the same universality class as the cell automaton models. Carlson *et al.* [10] explained why certain open driven systems organize to a critical state. Their key result is that the continuum limit of a certain self-organizing model has a diffusion coefficient which is singular at the critical point. The anomalous diffusion described in Ref. [10] is associated with the collective behavior of many avalanches in the dynamical equilibrium state. We investigate in this Letter the dynamics of *individual avalanches off criticality,* and show that it is also governed by an anomalous diffusion equation.

First, we describe briefly the dynamics of the original BTW sandpile model [1]. To each site of a hypercubic lattice an integer state variable is assigned. A site is called activated if its state variable exceeds a prescribed threshold value. At each time step, the state variable of each activated site decreases by a *fixed amount,* which is distributed *uniformly* between its *nearest neighbors,* increasing the value of their state variable. A variant of this dynamics was introduced by Zhang [7], where the state variable is continuous, and an activated site distributes its *whole content* between its neighbors. It is commonly believed that these two models belong to the same universality class. We are interested in this paper in the time evolution of *avalanches,* which are the connected clusters of sites with state variable near the threshold value.

Without the threshold condition these dynamics could be interpreted as discretized versions of a simple diffusion process (see, e.g., Press *et al.* [11]). As the threshold condition depends only on the local value of the state variable, it is appealing to look for an *anomalous diffusion equation* with a diffusion coefficient depending only on the state variable, whose discretization would give back the original dynamics. Such a continuum equation would hopefully allow an analytic treatment, providing new information on the original discrete models. Of course, such a continuum description cannot reproduce the microscopic details of the dynamics, only the large-scale features. But our object of study is the time evolution of the avalanches, which are large-scale objects, so our guess is that the continuum description might prove useful. We note that, e.g., the Hwa-Kardar equation [8] arose from similar ideas, namely, to add nonlinear terms to the ordinary diffusion equation in order to get a continuum version of the SOC threshold dynamics.

Let us take the most general anomalous diffusion equation satisfying the above-mentioned condition:

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} = \mathrm{div}[D(\rho(\mathbf{r},t))\mathrm{grad}\rho(\mathbf{r},t)] = \Delta\Gamma(\rho(\mathbf{r},t)), \quad (1)$$

where $\rho(\mathbf{r},t)$ is the state variable which we take to be continuous and shall call *density* from now on, $D(\rho)$ is the density-dependent diffusion coefficient, and $\Gamma(\rho)$ is given by

$$\Gamma(\rho) = \int_0^\rho D(\rho')\,d\rho'. \quad (2)$$

Discretizing the Laplacian on a hypercubic lattice to lowest order in the lattice spacing, and discretizing the time variable as well, the following dynamics results for the discrete version of Eq. (1): The change (in one time step) of the density ρ at a given site is proportional to the difference between the value of $\Gamma(\rho)$ at that site and the average of $\Gamma(\rho)$ over its nearest neighbors. In the case of normal diffusion, $\Gamma(\rho)$ equals ρ up to a constant factor (the diffusion coefficient).

In order to build in the threshold behavior into the above dynamics, $\Gamma(\rho)$ should be chosen appropriately. Namely, $\Gamma(\rho)$ must have the form

$$\Gamma(\rho) = f(\rho)\Theta(\rho-\rho_c), \quad (3)$$

where Θ denotes the Heaviside step function, ρ_c is the threshold value of the density, and $f(\rho)$ is a coefficient characterizing the model at hand (the amount by which the density of an activated site decreases in one time step). For example, $f(\rho) = \mathrm{const}$ for the BTW model, and $f(\rho) = \rho$ for the Zhang model.

The anomalous diffusion coefficient may be calculated from Eq. (2): it is the derivative of $\Gamma(\rho)$ with respect to ρ. Since the Heaviside step function is discontinuous, we regularize it in order to get a meaningful $D(\rho)$ from Eq. (3). The simplest choice is

$$\Theta_{\mathrm{reg}}(\rho-\rho_c) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{\rho-\rho_c}{\epsilon}\right), \quad (4)$$

where ϵ is a small, positive regularization parameter.

The most important feature of the resulting diffusion coefficient is that it has a *sharp pronounced peak* at the threshold value ρ_c. This results in the following qualitative picture. At a given time, one may distinguish three different spatial regions: a critical region, where $\rho \approx \rho_c$, a supercritical region, where $\rho > \rho_c$, and a subcritical one, where $\rho < \rho_c$. In the subcritical and supercritical regions the diffusion coefficient is very small compared to that of the critical region. On the supercritical-critical boundary the diffusion coefficient grows steeply, and there is a large flux which enters the critical region. In the critical region the diffusion coefficient is very large, thus any inhomogeneity disappears "instantaneously." Consequently, the inflow flux is transported through the critical region to the critical-subcritical boundary, where it enters the subcritical region. The density at the supercritical-critical boundary layer decreases until it reaches the threshold value ρ_c, when the layer becomes part of the critical region. As a result, the boundary moves towards the supercritical region, until this later is completely absorbed by the critical one. The behavior of the other boundary is similar; the only difference is that the direction of the flux is opposite. The critical region grows steadily in this direction as well. An open finite system organizes itself into a critical state where the density reaches the threshold value ρ_c in the whole region, resulting in the "supersensitive" behavior where all perturbations relax quickly (SOC). We stress that we consider the time evolution of an arbitrary initial state, i.e., the self-organizing process towards criticality, not only local perturbations in the SOC state.

The time evolution of the system is determined by two distinct time scales: In the critical region there is a quick diffusion process with characteristic time τ_1, and in the off-critical region there is a much slower relaxation with time scale τ_2: $\tau_1 \ll \tau_2$. The spreading of the critical region takes place on a medium time scale τ:

$$\tau_1 \ll \tau \ll \tau_2. \quad (5)$$

This fact allows us to develop an effective theory describing the time evolution of the avalanches by using an adiabatic approximation. In the adiabatic approximation, we take the limits $\tau_1 = 0$ and $\tau_2 = \infty$. This means that in the critical region the relaxation process is instantaneous, while in the off-critical region there is no diffusion at all. In this approximation, we derived first-order ordinary differential equations, which govern the time evolution of the critical region [12].

We have studied numerically the solutions of Eq. (1) in one dimension with the regularized diffusion coefficient, with various initial conditions. We have found that the time evolution of the critical region is composed of two different motions: *spreading and creeping.* The critical region spreads following a power law,

$$l(t) \sim t^p, \quad (6)$$

where l is the linear size of the critical region, and the exponent $p = \frac{1}{3}$. This exponent differs from the "dynamical exponent," which relates the linear size of an avalanche to time in the critical state. The exponent p characterizes the buildup process of the critical region off SOC. Creeping means that the center of the critical region, defined as the point where the diffusion coefficient takes on its maximum, shifts continuously (except when the symmetry of the problem forbids it). This creeping obeys again a power law,

$$x_0(t) \sim t^s, \tag{7}$$

where x_0 denotes the distance of the center from its initial position, and the exponent $s=\frac{2}{3}$. The above power-law behaviors and the exponent values were insensitive to the different initial conditions, and to the detailed form of the diffusion coefficient (i.e., the regularization). Moreover, the adiabatic approximation alluded to before predicts the same power laws.

We have compared the above results with the behavior of both discrete automaton models (BTW and Zhang) following the procedure of Lebowitz, Presutti, and Spohn [13]. One prepares a statistical ensemble of random initial conditions for the cellular automaton, such that the ensemble-averaged state variable coincides with the macroscopic initial density profile. Each member of the ensemble evolves according to the discrete automaton algorithm. The time evolution of the ensemble-averaged state variable (identified with the macroscopic density profile) agrees exactly with the solution of Eq. (1). Starting from a linear initial profile, we observed both the spreading and the creeping of the critical region. They obey the power laws [Eqs. (6) and (7)] with empirical exponents $p=0.33288\pm0.008$ and $s=0.65\pm0.018$. Moreover, the shape of the empirical diffusion coefficient evaluated from the simulation data is identical to the predicted one, giving a direct proof of the validity of our approach.

We plotted in Fig. 1(a) the time evolution of the density profile in one dimension, for a linear initial density distribution, obtained by numerical solution of Eq. (1). The diffusion coefficient was

$$D(\rho)=D_0\left[\arctan\left(\frac{\rho-\rho_c}{\epsilon}\right)+\frac{\pi}{2}+\frac{\epsilon\rho_c}{\epsilon^2+(\rho-\rho_c)^2}\right], \tag{8}$$

whose shape is shown in Fig. 2(a) (the narrow asymmetric peak belonging to $t=0$). This diffusion coefficient corresponds to the Zhang dynamics, where $\Gamma(\rho)=\rho\times\Theta_{\rm reg}(\rho-\rho_c)$. Figure 2(a) shows the time evolution of the diffusion coefficient's profile, exhibiting clearly the creeping of the critical region. Figures 1(b) and 2(b) show the above-mentioned power-law behaviors [see Eqs. (6) and (7)].

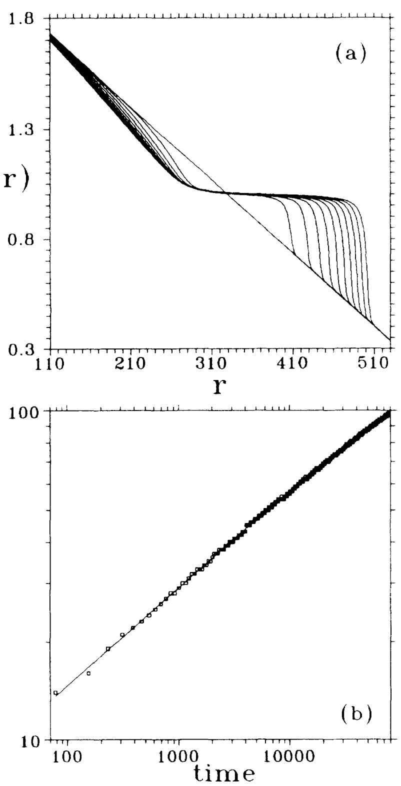

FIG. 1. (a) Time evolution of the density profile in a one-dimensional system, where $L=661$. The initial condition is the linear density distribution; the diffusion coefficient is given by Eq. (8), $D_0=0.032$, $\epsilon=3.3$, $\rho_c=1$, $r(\rho=\rho_c,t=0)=330$, the curves plot the instants $t=0$, 7735, 15468, 23201, 30934, 38666, 46399, 54131, 61863, 69596, and 77328. (b) The growth of the critical region width l on a log-log scale. The slope is $p=0.32\pm0.02$.

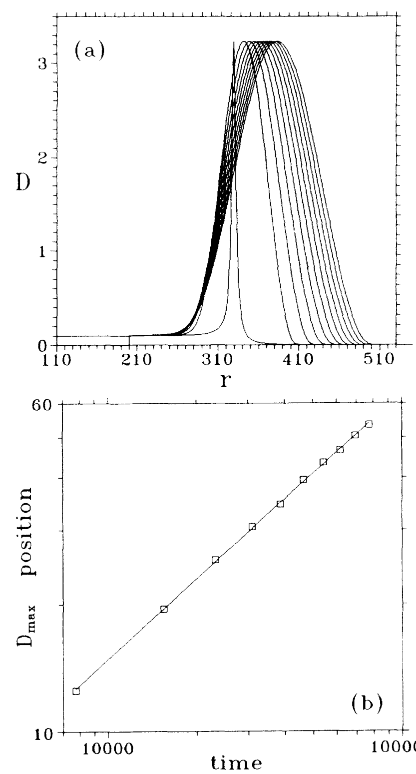

FIG. 2. (a) Time evolution of the diffusion coefficient profile measured in the same system and at the same instants as in Fig. 1(a). The diffusion coefficient at $t=0$ is given by Eq. (8). (b) The shift of the diffusion coefficient peak. The power-law fit gives the exponent $s=0.64\pm0.016$.

In summary, we have studied an anomalous diffusion equation obtained as a continuum limit of the BTW and Zhang dynamics. We guess that in those systems, where the density-dependent diffusion coefficient has a sharp maximum at some threshold value, the emergence of SOC is a generic phenomenon. We have shown analytically and numerically in the one-dimensional case that the time evolution of the linear size of the critical region follows a power law, with exponent $\frac{1}{3}$. Moreover, we have found that the center of the critical region creeps continuously, obeying again a power law with exponent $\frac{2}{3}$. These results are quite universal, insensitive to the detailed form of the initial density distribution and the diffusion coefficient, provided the latter has a sharp pronounced peak at some threshold value, and we are far from the fully developed SOC state. We note here that recent publications [14,15] show examples where the diffusion coefficient has a maximum with respect to the density. We hope that our work can lead to a better understanding of the processes by which the SOC state is built up, and will extend the circle of real systems where SOC occurs.

We wish to thank Géza Tichy (Eötvös University) for discussions, and Joachim Krug (IBM, T. J. Watson Research Center) for drawing our attention to Ref. [13], and suggesting a practical method. This work has been partially supported by OTKA under Grant No. 2091.

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