Vortex Rings and Finite-Wave-Number Superfluidity near the ^{4}He λ Transition

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The superfluid response to a perturbation of wave number k is calculated using a vortex-ring theory of the ^{4}He λ transition. Rings of diameter greater than k^{-1} cannot follow the perturbation, and this leads to a finite-size broadening of the superfluid density near T_λ. The vortex theory provides a new perspective on the Feynman-Onsager proposal that rotons are vortex rings of smallest size.

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The ^{4}He superfluid λ transition is known to be in the same universality class as the three-dimensional XY model of magnetism. Recently, a vortex-ring theory [1,2] of the XY model has been successful in calculating the helicity modulus of the spins [3] (a quantity analogous to the ^{4}He superfluid density), in agreement with Monte Carlo simulations [4,5]. In this Letter the vortex theory is applied to the ^{4}He superfluid transition. The superfluid density at finite wave number is calculated by considering the response of a vortex ring to a spatially modulated superflow. The formulation in terms of topological excitations allows some new insights into the underlying basis of the transition.

The vortex-ring theory is based on two simple ideas [6]: (a) The rings are dipoles which orient in the presence of an external flow such that their net current reduces the total flow, reducing the superfluid density, and (b) the energy of a large ring is reduced by the screening effect of small rings being oriented by the flow of the large ring. These ideas lead to real-space recursion relations [1-3] for the XY coupling constant K and vortex fugacity y,

$$\frac{\partial(1/K)}{\partial l} = -\frac{1}{K} + A_0 y\,, \tag{1}$$

$$\frac{\partial y}{\partial l} = \{6 - \pi^2 K[\ln(K^{-\theta}) + 1]\}y\,, \tag{2}$$

$$K_r = \lim_{l\to\infty} K e^{-l}. \tag{3}$$

In these equations l is a dimensionless length, $l = \ln(a/a_0)$, where a is the average diameter of a ring and a_0 is the "bare" vortex core diameter (the lattice constant in the XY model). K_r is the observable renormalized coupling constant, the helicity modulus, and is related to the superfluid density ρ_s at temperature T by $K_r = \hbar^2\rho_s a_0/m^2 k_B T$, with m the mass of a helium atom. The presence of noncircular rings is taken into account in Eq. (2) using the Flory scaling proposal of Shenoy [2] for the effective core size a_c; $a_c/a = K^\theta$, with the exponent θ taken to be the Flory value $\theta = 3/(D+2) = 0.6$ in $D=3$ dimensions. Although initially just an ansatz, there is now more compelling evidence in favor of this value for the exponent [7]. The iteration of Eqs. (1)-(3) starts at the scale a_0 from initial values $K_0 = K_r\rho_s^0/\rho_s$ and $y_0 = e^{-\pi^2 K_0 C}$, where ρ_s^0 is the bare superfluid density from excitations other than vortices, and C is a constant related to the vortex core energy. In the 3D XY model C is known to have the value $\frac{4}{3}$ [8]. The value of the constant A_0 is determined to be $A_0 = 24.4$ by requiring that the critical value of K_0 match the value known for the XY model [3,5].

By iterating these equations to length scales up to the correlation length $\xi = a_0/K_r$, a power-law phase transition is found of the form $K_r \sim (K_0 - K_{0c})^\nu$, with the exponent ν in agreement with Shenoy's prediction, $\nu = 0.6717$. The correlation length is essentially the diameter of the largest rings being thermally excited. Truncation of the iterations at a finite length scale leads to a finite-size broadening of the phase transition, with results in agreement with both Monte Carlo simulations on finite lattices and with the predictions of finite-size scaling [3-5].

To apply this theory of the XY model to the helium λ transition, it is necessary to determine the core energy of the helium vortices, which is not a universal quantity. This is accomplished by adjusting the core energy constant C until the magnitude of the superfluid density in the critical region $T \approx T_\lambda$ matches the experimental value [9], $\rho_s/\rho = 2.40 t^{0.6717}$, where $t = (T_\lambda - T)/T_\lambda$. This requires an assumption about the bare superfluid density ρ_s^0, since the recursion relations yield only the quantity ρ_s/ρ_s^0 as a function of temperature. From the approximate solutions of Shenoy [2] for ρ_s/ρ as a function of ρ_s^0 and C, and from more accurate numerical solutions, it quickly becomes apparent that the experimental critical amplitude can be matched only with $\rho_s^0/\rho \simeq 1.0$ for values of C in the neighborhood of $\frac{4}{3}$. To get values of ρ_s^0/ρ in the range of 0.1-0.3 that might be expected from the λ-point scenario of Ref. [1] (where roton excitations were assumed to be excitations different from the vortex rings) would require unreasonably large values of C, an order of magnitude larger than the XY value. It is concluded that only phonon excitations should be included in the bare superfluid density, and since they make a negligible contribution to ρ_s even near T_λ, we set $\rho_s^0/\rho \equiv 1.0$ in evaluating the renormalized superfluid density. The detailed fit to the experimental critical amplitude then yields a core energy constant $C = 1.03$, about 30% smaller than the value for XY vortices. This is quite consistent with the experimental results found in two-dimensional helium films [10], where the core energy constant was measured to be about a factor of 2 smaller than that for XY vortices. In this scenario roton excitations must be identified with the vortex rings of smallest size, those of diameter a_0 where the recursion of Eqs. (1)-(3) begins.

With this value for C the helium transition is found to occur at a critical value $K_{0c}=0.2772352$, about 20% higher than the XY result [5]. This corresponds to a critical temperature [11] given by physical constants and the fundamental length scale a_0,

$$T_\lambda = \hbar^2 \rho_s^0 a_0 / m^2 k_B K_{0c} \,. \tag{4}$$

Inserting the known value $T_\lambda = 2.172$ K yields $a_0=2.27$ Å. This result is consistent with the expectation that the bare core size be of the order of the interatomic spacing of liquid helium, $d=3.6$ Å, and is also consistent with experimental determinations of the core size [12]. It is also quite reasonable for the size of a roton, which is thought to be a microscopic excitation involving the flow of just a few neighboring atoms [13]. The roton energy at T_λ can be identified as the energy of the smallest ring, $\Delta/k_B = \pi^2 K_{0c} C T_\lambda = 6.1$ K, which compares well with experimental values of 5.8 K from neutron scattering [14] and 7.5 K from Raman scattering [15].

The idea of rotons as a modified form of the smallest vortex rings was first proposed by Onsager and Feynman [16], and this concept has been extended more recently [17]. Although the atomic nature of the liquid will change the energy and dispersion of the excitation from that of a macroscopic vortex ring, the only feature needed for the renormalization scheme outlined above is that the excitation have a dipole moment that can be oriented by an applied flow field. Feynman and others have demonstrated that roton excitations have just this property [18]. The detailed microscopic nature of the rotons does not affect the present calculations near T_λ because only the largest rings control the behavior at the critical point.

The dynamics of the superfluid transition can be investigated by studying the response of the vortices to perturbing flow fields, similar to the methods used by Ambegaokar *et al.* [19] in two dimensions. The flow is taken to have the form $\mathbf{v}_s = \mathbf{v}_{s0} \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$, where $\mathbf{k}$ is oriented to be perpendicular to $\mathbf{v}_{s0}$, since it is the transverse component that is of interest. For general values of ω and $\mathbf{k}$ it is necessary to solve a Fokker-Planck equation [20] for the altered distribution function of the vortex rings. A simpler calculation, however, is to take the limit $\omega \to 0$ and find the wave-number-dependent superfluid density $\rho_s(k)$, which can be compared with a recent Monte Carlo simulation [21] of this quantity.

The response of a circular vortex ring to a spatially varying field can be computed by analogy with the torque exerted on a current loop in a magnetic field. The ring of diameter a is taken to be centered in the x-y plane, with the vortex core located at polar coordinates $(a/2,\phi)$. The flow $\mathbf{v}_{s0}$ is taken to be in the y-z plane inclined at an angle θ from the z axis, and $\mathbf{k}$ is taken along the x axis. The torque on the ring is then

$$\begin{aligned}\mathbf{N} &= \hat{\mathbf{x}} 2\pi \frac{\hbar}{m} \rho_s^0 \left(\frac{a}{2}\right)^2 v_{s0} \sin\theta \int_0^{2\pi} e^{ik(a/2)\cos\phi} \sin^2\phi \, d\phi \\ &= \hat{\mathbf{x}} g(k) p v_{s0} \sin\theta \,, \end{aligned} \tag{5}$$

where p is the momentum of the vortex ring (along the z axis), $p = 2\pi^2 (\hbar/m) \rho_s^0 (a/2)^2$, and $g(k) = 2J_1(ka/2)/(ka/2)$, where J_1 is the Bessel function. The change in the energy of the ring is then

$$\delta U = \int |\mathbf{N}| d\theta = -g(k) \mathbf{p}\cdot\mathbf{v}_{s0} \,. \tag{6}$$

The physical interpretation of this result is quite simple: At long wavelengths $ka \ll 1$ the response function $g(k) \to 1$, and Eq. (6) is the usual expression for the energy change in a homogeneous flow field. When the wavelength approaches the ring diameter, however, $g(k)$ decreases, dropping to zero at $ka=7.6$ and oscillating with small amplitude about zero for larger k. In this regime the effect of the flow averages to zero across the diameter of the ring. Since Eq. (6) has the same form as used in computing the ring polarizability in Ref. [1] it is straightforward to deduce that the susceptibility at finite wave number $\mu(k)$ is found from

$$\mu(k) = \frac{\rho}{\rho_s(k)} = 1 + \int_{a_0}^{\infty} \frac{\partial \mu(0)}{\partial a} g(k) \, da \,, \tag{7}$$

where $\mu(0) = K_0/K_r$ is the susceptibility calculated from Eqs. (1)-(3).

Calculating $\rho_s(k)$ from Eq. (7) gives a broadened phase transition, with ρ_s remaining finite at T_λ. This is essentially a finite-size broadening, since $g(k)$ cuts off the

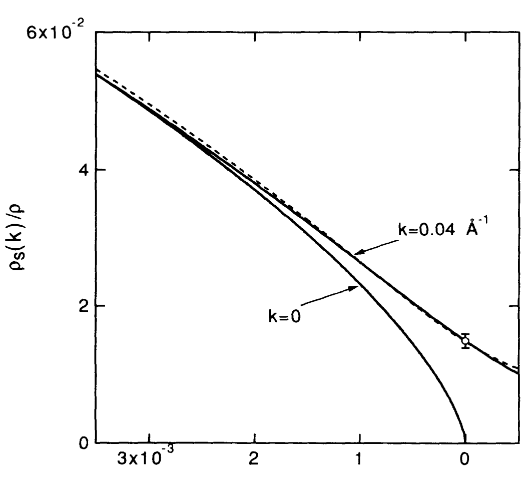

FIG. 1. Wave-number-dependent superfluid density vs reduced temperature t. The solid curves are calculated from the vortex theory [Eq. (7)]. The dashed line is the interpolation formula of Eq. (8), and the data point indicates the Monte Carlo result at $t=0$ from Ref. [21].

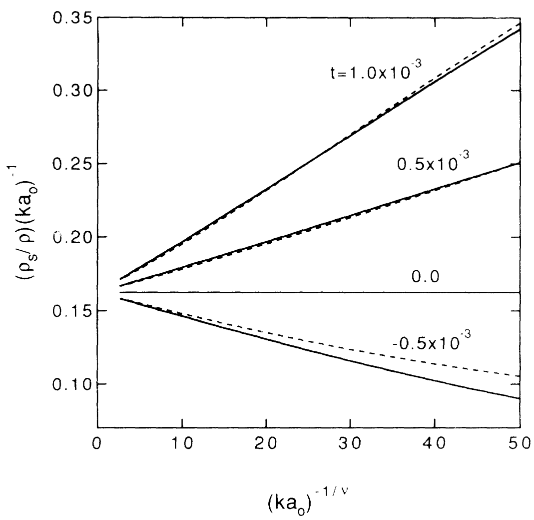

FIG. 2. Scaling plot of $(\rho_s/\rho)(ka_0)^{-1}$ vs $(ka_0)^{-1/\nu}$. The solid lines are the vortex theory, while the dashed lines are Eq. (8).

recursion relations at the length scale $\sim k^{-1}$. The largest rings cannot respond to the spatially varying flow field, and hence do not contribute to the reduction of the superfluid density. Using the above circular-ring expression for $g(k)$ to evaluate ρ_s at T_λ, the universal quantity K_r/ka_0 is found to be 0.097. This is close to the early calculation of Ferrell *et al.* [22] which predicts $1/\pi^2=0.101$, but does not agree with the recent Monte Carlo simulation of Cha *et al.* [21], which gives a value of 0.046 for this quantity. It appears that the circular-ring approximation for $g(k)$ is not entirely accurate. It is plausible that as the probe wavelength approaches, the ring diameter $g(k)$ could be affected by the smaller scale twists and turns of the actual distorted rings; but in any case the response must still finally drop to zero for large enough k. To model this in a fashion similar to the finite-size calculation [3], an alternative step-function form for $g(k)$ has been employed, taking $g(k)=1$ up to $ka=8.6$, and then zero for larger values of ka. This cutoff point is chosen to give agreement with the Monte Carlo results, and is slightly higher than the zero of the Bessel function $g(k)$. Figure 1 shows the results of calculating ρ_s from Eq. (7) using the step-function form of g, for the values $k=0$ and 0.04 Å^{-1}. The Monte Carlo prediction at T_λ is shown as the data point, and the dashed line is the interpolation formula conjectured in Ref. [22],

$$\frac{\rho_s(k)}{\rho}=1.51\{t+[t^2+(2k\xi_0)^{2/\nu}]^{1/2}\}^{\nu}, \qquad (8)$$

where $\xi_0=0.118$ Å is adjusted to match the value at T_λ.

Figure 2 shows the k dependence of the superfluid density, plotted in the finite-size scaling form [3,4] of $(\rho_s/\rho)(ka_0)^{-1}$ vs $(ka_0)^{-1/\nu}$. It is only precisely at T_λ that ρ_s becomes a linear function of k, shown by the zero slope at $t=0$. It is this dependence on k that gives rise to the anomalous dispersion and critical damping of second sound, as first shown in Ref. [22]. The dashed lines in Fig. 2 are the interpolation formula of Eq. (8), and again there is general agreement between this form and the vortex results, at least below T_λ.

In summary, a vortex-ring theory of the ^{4}He λ transition provides evidence supporting the Feynman-Onsager proposal that rotons are the limit of the smallest vortex rings. The vortex theory is able to reproduce the wave-number-dependent broadening of the transition known from earlier studies. Although the connection between finite size and finite wave number has been previously noted [23], the topological formulation now provides a clear physical picture of the mechanism involved. It should now be possible to extend these results to finite frequencies and study the dynamics of the transition, such as the critical damping of first and second sound by the vortices.

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ideal Bose gas. In the theory of Bose condensation the transition occurs when the de Broglie wavelength $\lambda = (h/3mk_B T)^{1/2}$ becomes comparable to the interparticle spacing d, with a critical value $\lambda_c = 2.0d$ when $T = T_c$. Inserting Eq. (4) for T_λ into the de Broglie wavelength gives a different expression for the critical value, $\lambda_c = 2.7d(d/a_0)^{1/2}$. The two expressions do happen to give critical temperatures in the same range, but this is only because the strong interactions between the particles give a vortex core size a_0 that is nearly the same as the interatomic spacing d. Liquid helium is not at all an ideal gas, and the mechanism underlying the superfluid transition is quite different from the Bose condensation picture that is often cited in textbooks and popular articles.

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