Evolution of River Networks

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(Received 22 October 1991)

Why does a sheet of water flowing over an initially featureless surface spontaneously form a river network? To address this question, we construct a simple model which enables us to examine the shape and stability of individual river channels. We compare predictions for the geometry of fluvial channels with experimental data. In addition, we construct a lattice model which allows us to look at large-scale features of river networks and calculate their scaling relations.

PACS numbers: 68.70.+w, 92.40.Fb, 92.40.Gc

If fractals are the geometry of nature, one must still ask how nature produces them. Branched river networks are among nature's most common patterns, spontaneously producing structure over a huge range of length scales [1]. At the heart of this problem is the question of why and how individual river channels are formed. This Letter presents a nonlinear model which describes the evolution of an arbitrary initial landscape covered by a distribution of water, and has the goal of understanding instabilities which lead to the coalescence of the water into channels and, later, river networks.

Rivers have been studied extensively by a wide variety of researchers with an equally wide variety of techniques and goals. Geomorphologists have found scaling relationships among various combinations of basin statistics from field data, such as drainage density and branching ratios [2]. Hydrologists have likewise extracted power laws for channel parameters such as width, depth, velocity, and slope as functions of total channel discharge [3]. Other investigators have examined the shape [4–6] of individual equilibrium channels in erodible material while some have constructed models for the evolution of an entire drainage network [7–9]. However, we are unaware of a previous approach which predicts the shapes spontaneously formed when water flows over an initially featureless surface and allows one to answer questions about selection and stability. We differ from previous authors because we do not try to find closed equations for height of soil alone; we treat water explicitly as well. The aim is to find the simplest model which reveals the essential features of river formation.

We pose our model in terms of two scalar fields, $b(x,y,t)$ and $d(x,y,t)$, where $b(x,y,t)$ is the height of soil above some arbitrary horizontal level, $d(x,y,t)$ is the depth of water flowing over the soil, x and y are spatial coordinates, and t is time. It will be convenient to introduce an auxiliary field $s(x,y,t)\equiv b(x,y,t)+d(x,y,t)$ which defines the overall surface of the land plus water. The time evolution of the system is given by

$$\frac{\partial b}{\partial t}=-\frac{P}{P_0}v[1+(\nabla b)^2]^{1/2}+\nu\nabla\cdot(vd\nabla b)\,, \tag{1a}$$

$$\frac{\partial d}{\partial t}=-\nabla\cdot(\mathbf{v}d)+f_p\,, \tag{1b}$$

$$\mathbf{v}=-\frac{(2gd)^{1/2}\nabla s}{[1+(\nabla s)^2]^{1/2}},\quad P=\frac{\rho g d}{1+(\nabla s)^2}\,. \tag{1c}$$

Equation (1a) gives the time evolution of the soil field. The first term on the right represents erosion, which is assumed to be proportional to a product of water pressure P, velocity v, and a constant P_0 which indicates the erodibility of the soil. Erosion takes place normal to the bottom surface, necessitating the geometrical factor [10] $[1+(\nabla b)^2]^{1/2}$. The second term on the right of Eq. (1a) represents turbulent diffusion and accounts for a redistribution of soil along the bottom due to turbulent eddies in the flow. Although ν may be small, we will see that this term plays a crucial role in stabilizing river channels. Conservation of water is expressed by Eq. (1b), in which the first term on the right is the convective flux of water, and the second is a spatially varying vertical flux of water from precipitation. Precipitation is included for generality but will frequently be omitted in the following analysis. It should be emphasized that soil is not treated as a conserved quantity; once leached into the water, it is transported off to the ocean. Thus, we are working in the limit where erosion is sufficiently slow that the capacity of the water to carry sediment is never exceeded [11].

Given the profile of the land b and the depth of the water d, one must deduce the speed at which the water moves. The vertical component of the velocity is assumed to be negligible, and the z dependence of the horizontal velocity components is likewise neglected. We require the velocity and pressure of the water to obey three conditions. First, one has Bernoulli's law, $P=\rho g d-\rho v^2/2$. Second, for small surface gradients, we want water to flow downhill according to $\mathbf{v}\sim-\nabla s$ [12]. Third, for large velocities, as occur when water is in free fall, we want the pressure to vanish. The simplest set of equations obeying these conditions is Eq. (1c). With these expressions for the velocity and pressure, the system defined by Eq. (1) is closed.

An important property of this system is that it admits steady-state solutions, which descend a gradient of slope α at a constant rate r without changing shape. These solutions are of the form

$$\begin{aligned} d(x,y,t)&=d(x)\,,\\ b(x,y,t)&=-d(x)-\alpha y-rt\,. \end{aligned} \tag{2}$$

Working in the limit $\alpha\to 0$, which is appropriate for actual rivers, defining the characteristic length $d_0=P_0/\rho g$, time $t_0=(d_0/2g\alpha^2)^{1/2}$, and expressing all variables in

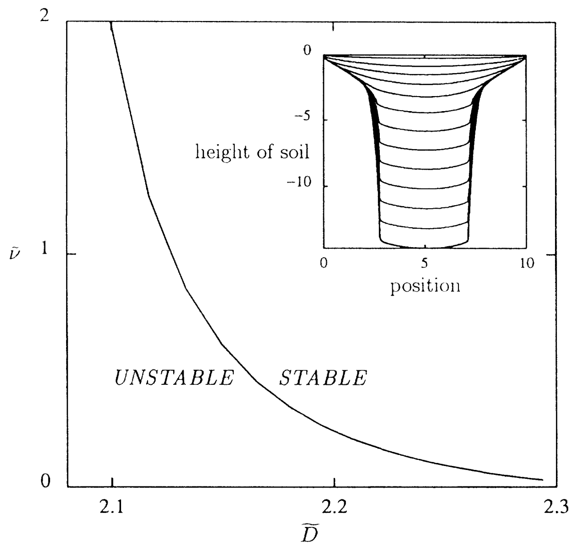

FIG. 1. Neutral stability curve in the $\tilde{\nu}$-$\tilde{D}$ parameter plane, where $\tilde{\nu}$ and $\tilde{D}$ are the scaled dimensionless turbulent diffusion coefficient and maximum channel depth, respectively. Steady states above the line are stable while those below it are unstable. Inset: The time evolution of an arbitrary initial state which is shown adjusting its width to approach a steady state.

these units, one finds that

$$r=d^{3/2}[1+(d')^2]^{1/2}+\nu\frac{\partial}{\partial x}d^{3/2}\frac{\partial}{\partial x}d\,. \qquad (3)$$

For all values of $\nu>0$ and $r>0$ this equation admits a continuous family of physically acceptable solutions of the form $d(x)=r^{2/3}f(\tilde{x};\tilde{\nu},\tilde{D})$, where $\tilde{\nu}=\nu/r^{2/3}$, $\tilde{x}=x/r^{2/3}$, indexed by $\tilde{D}=D/r^{2/3}$, with D the maximum channel depth.

Notice that when the erosion rate r is small, the effective diffusion constant $\tilde{\nu}$ becomes large. Diffusion stabilizes channels, and since wide shallow channels descend more slowly than narrow deep ones, shallow ones should be preferred [13]. More generally, we ask how an arbitrary initial state evolves into a steady-state solution, dynamically selecting a particular profile out of the continuous family of possible choices [14]. If one considers the problem of an infinitely wide sheet of water of depth D over a flat plane, one finds the first unstable mode to have wavelength $\lambda=2\pi\sqrt{2\nu D/3}$. However, this wavelength is larger than the channel width that is actually selected by time-evolving [15] arbitrary initial states according to Eq. (1). By doing a two-dimensional numerical stability analysis of Eq. (1) about the exact solutions of Eq. (3), one finds that only for certain values of $\tilde{\nu}$ and $\tilde{D}$ is the solution stable [16]. Figure 1 shows the neutral stability curve in the $\tilde{\nu}$-$\tilde{D}$ parameter plane. Steady-state solutions which lie above this curve are stable, and those below it, unstable. An initial profile that lies in the unstable region moves toward the neutral stability curve as it is time evolved; a sequence of successive states is shown as an inset in Fig. 1.

Although we have consistently favored simplicity over realism in our model, it is reasonable to ask how well our results compare with field data for real rivers. In Tables I and II, we summarize the results for various exponents [17]. Channel geometry relations (Table I) for w, D, and v as functions of the total channel discharge Q, which has units of volume water per time, agree fairly well with the empirical results, especially those of Ackers. The scaling of mean water velocity with channel depth and land slope is displayed in Table II. The exponents for our model are analytical consequences of Eqs. (1) and (3). Given that the exponents from field data are not mutually consistent, agreement with our model is satisfactory.

The computer time required to study Eq. (1) on a large scale is formidable. In order to learn about larger-scale features of river networks, we have abstracted from our previous work a simple lattice model [9,18] which proceeds according to the following rules.

(1) At each site of a lattice, we specify two integers, one corresponding to the height of land, the other to the height of water.

(2) A lattice site is chosen randomly, and if the surface height (water plus land) is lower on a neighboring site, water units are moved to bring the surfaces as close to even as possible.

(3) For each water unit transported out, a unit of land is dissolved away—but only if the land is lower at the destination site.

(4) Additional water falls on a site as precipitation at random intervals.

The channel networks developed by the lattice model

TABLE I. Scaling relations of width w, maximum depth D, and mean velocity v as functions of channel discharge Q are displayed for a variety of empirical investigations, and for the results of our Eq. (3).

Source	$w \sim Q^b$	$D \sim Q^f$	$v \sim Q^m$
Leopold and Maddock, Ref. [3] (downstream)	$b=0.50$	$f=0.40$	$m=0.10$
Leopold and Maddock (at a cross section)	$b=0.26$	$f=0.40$	$m=0.34$
Leopold and Miller, Ref. [26]	$b=0.25$	$f=0.41$	$m=0.33$
Ackers, Ref. [27]	$b=0.42$	$f=0.43$	$m=0.15$
Present model	$b=\frac{2}{5}$	$f=\frac{2}{5}$	$m=\frac{1}{5}$

TABLE II. The scaling of the velocity v with land slope α and channel depth D postulated in Eq. (1c) is compared with other authors' relations.

Manning, Ref. [28]	$v \sim D^{2/3}\alpha^{1/2}$
Lacey, Ref. [29]	$v \sim D^{1/2}$
Smith and Bretherton, Ref. [7]	$v \sim \alpha$
Present model	$v \sim D^{1/2}\alpha$

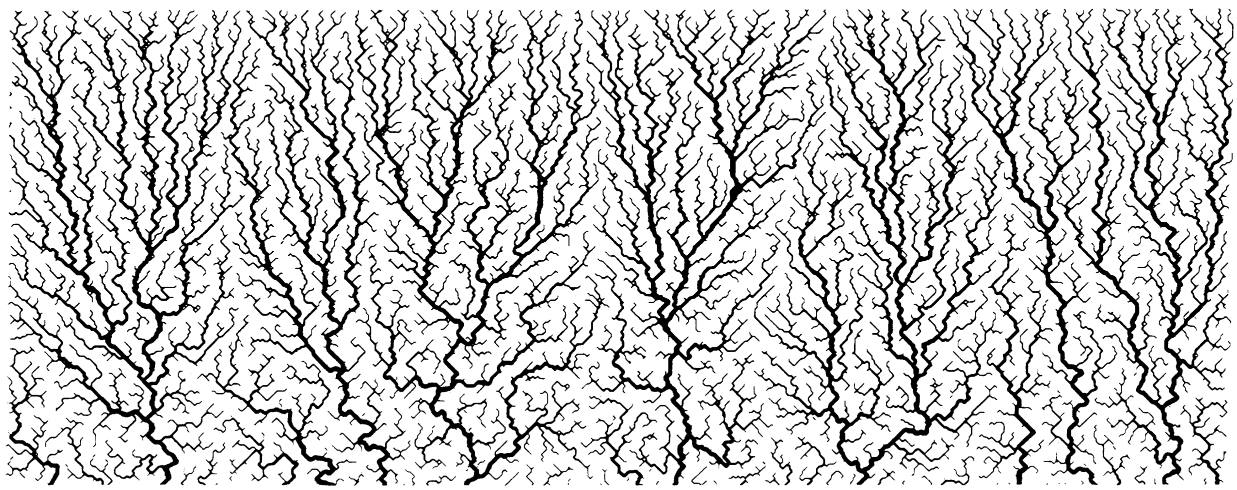

FIG. 2. Channel network generated by the lattice model of the river evolution equations. This run was done on an initially flat 200×500 square lattice over a time period of 2×10^9 iterations. Boundary conditions are periodic on the sides, a rigid barrier at the top, and an outflow boundary at the bottom. The width of the lines is proportional to the time-averaged flow through each point.

described above follow the qualitative stages of network evolution postulated by Glock [19]: initiation, elongation, elaboration to maximum development, and abstraction (reduction as smaller streams vanish due to screening effects of larger neighbors). The time-average flow [20] through the network is shown in Fig. 2.

Smart [21] has collected data on a drainage network in the eastern coal fields region of Kentucky and reported these data using Shreve's link magnitude formalism [22]. To compare with our data, let N_u be the number of links [23], of magnitude u, where the link magnitude is defined

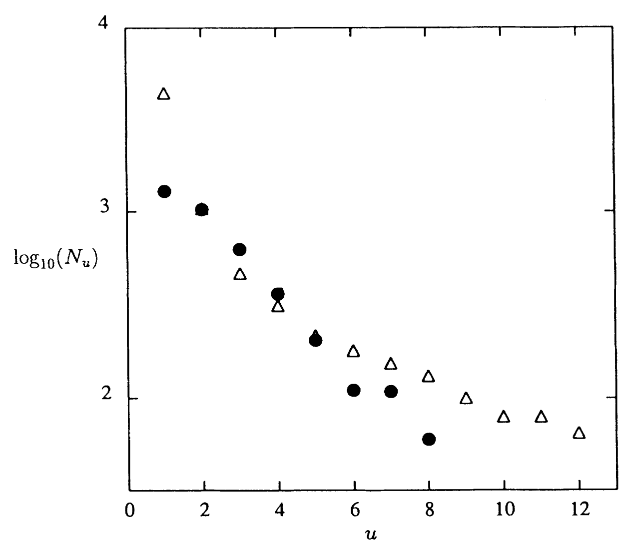

FIG. 3. Comparison of $\log_{10} N_u$ vs u for computer-generated channel network (●) and for a natural network in Kentucky (Δ), Ref. [21], where N_u is the number of links, or stream segments, of magnitude u. The link magnitude u is defined by the integer part of the logarithm of the flow.

by the integer part of the logarithm of the flow. Figure 3 shows a plot of $\log_{10}(N_u)$ vs u for Smart's data and for the average of three separate runs of our lattice model. There is a similarity between the two data sets, although neither is exponential to better than 10%.

The models presented here make many definite predictions about the selection and stability of river channels and networks. However, much remains to be understood. We need to obtain measurements of the turbulent diffusion coefficient ν in order to make detailed comparisons with field and laboratory channels. In most cases, realistic comparison will only be possible when we extend the model so as to conserve soil and allow for its redeposition. In addition, we hope to derive the lattice model from the continuum equations, and to find the relation between our lattice model and others used in pattern formation problems such as sandpiles [24] and diffusion-limited aggregation [25].

M.M. is grateful for support from the Sloan Foundation. Computational resources were provided by the University of Texas System Center for High-Performance Computing. We thank D. Lathrop for suggesting this problem, and L. Tuckerman for a careful reading of the manuscript.

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[13] Actual rivers are typically 20–200 times wider than they are deep, but the redeposition of sediment is sufficiently important in most cases that we are unable to make the comparison quantitative.
[14] This selection process should be contrasted with that for the dendrite or Saffman-Taylor problems where surface tension reduces the solution space from a continuum to a discrete set. Here the singular perturbation of the turbulent diffusion term has expanded the solution space from a unique solution to a continuous family of solutions parametrized by the channel depth D. See, for example, J. S. Langer, Phys. Rev. A **33**, 435 (1986), or D. Bensimon *et al.*, Rev. Mod. Phys. **58**, 977 (1986).
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[17] The exponents b, f, and m for the present model are most simply calculated assuming that the discharge Q varies while holding the scaled variables $\tilde{\nu}$ and $\tilde{D}$ constant. It is probably more realistic to vary Q so as to move along the neutral stability curve in Fig. 1, obtaining the values $b=0.221$, $f=0.512$, and $m=0.256$. We postulate that these exponents are relevant for a neutrally stable channel which evolves as Q is decreased.
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