Vortex Solutions in the Weinberg-Salam Model

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We show that the Weinberg-Salam model has vortex solutions similar to semilocal strings for all values of the parameters. The stability of the solutions under small perturbations will depend on the parameters of the theory and, in particular, on the ratio of the Higgs boson mass to the Z boson mass.

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It is generally believed that the standard electroweak model of Weinberg and Salam [1] is free from topological defects. The reason is that the first homotopy group of $SU(2)_L \times U(1)_Y/U(1)_{em}$ can be shown to be trivial. However, this does not mean that the model is free from nontopological defects and it is the purpose of this Letter to exhibit one such defect.

A hint that there may be defects in the Weinberg-Salam model comes from Ref. [2] where it was shown that the symmetry breaking $SU(2)_{global} \times U(1)_{local} \to U(1)_{global}$ admits vortex solutions known as "semilocal" strings. But this is precisely the Weinberg-Salam model with the $SU(2)_L$ charge (g) set equal to zero. Here we ask the next logical question: What happens if g is nonzero?

In what follows, we shall first show that the semilocal string leads to an *exact* vortex solution even in the case when $g \neq 0$. We then examine the question of stability and argue that the solution is stable to small perturbations for large values of the Weinberg angle and small values of the Higgs boson mass. We plan to investigate the stability of the solution in other regions of parameter space in a separate publication [3].

The Lagrangian for the Weinberg-Salam model (ignoring the fermions) is [4]

$$L = L_W + L_B + L_\phi - V(\phi) , \quad (1)$$

where

$$L_W = -\tfrac{1}{4} G_{\mu\nu a} G^{\mu\nu a} , \quad (2)$$

$$L_B = -\tfrac{1}{4} F_{B\mu\nu} F^{B\mu\nu} , \quad (3)$$

$$L_\phi = |D_\lambda \phi|^2 \equiv |(\partial_\lambda - \tfrac{1}{2} i g \tau^a W^a_\lambda - \tfrac{1}{2} i g' B_\lambda)\phi|^2 , \quad (4)$$

$$V(\phi) = \lambda(\phi^\dagger \phi - \eta^2/2)^2 , \quad (5)$$

and ϕ is a complex doublet.

While the most direct way to exhibit the vortex solution would be to show that it satisfies the Euler-Lagrange equations of motion [3], it is not the most convenient. Here, instead, we will demonstrate the validity of the vortex solution by showing that it extremizes the energy. As we are interested in static solutions that have translational invariance in the z direction, we can set the zero and third components of the vector fields to zero and confine our attention to the x-y plane. Then the energy (per unit length) following from (1) is

$$E = \int d^2x [\tfrac{1}{4} G^a_{ij} G^a_{ij} + \tfrac{1}{4} F_{Bij} F_{Bij} + (D_j\phi)^\dagger (D_j \phi) + \lambda(\phi^\dagger\phi - \eta^2/2)^2] , \quad (6)$$

where $i,j = 1,2$.

The vortex solution that extremizes this energy functional is

$$\phi = f_{NO}(r) e^{im\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{Z} = \mathbf{A}_{NO} , \quad (7)$$

and $\mathbf{A} = 0 = \mathbf{W}^{\bar a}$ ($\bar a = 1,2$). Here

$$\mathbf{Z} \equiv \cos\theta_W \mathbf{W}^3 - \sin\theta_W \mathbf{B}, \quad \mathbf{A} \equiv \sin\theta_W \mathbf{W}^3 + \cos\theta_W \mathbf{B} , \quad (8)$$

and the subscript NO on the functions f and $\mathbf{A}$ [in (7)] means that they are identical to the corresponding functions found by Nielsen and Olesen [5] for the usual Abelian-Higgs string. The coordinates r and θ are polar coordinates in the x-y plane and $\tan\theta_W = g'/g$ defines the Weinberg angle. The integer m is the winding number of the vortex. In what follows, we shall restrict ourselves to the case $m = 1$.

To see that the solution described above indeed extremizes the energy, let us write

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix} , \quad (9)$$

$$T^1 \equiv \mathrm{diag}(-\cos 2\theta_W, 1) , \quad (10)$$

and $d_j \equiv (\partial_j + i\tfrac{1}{2} \alpha T^1 Z_j)$ with $\alpha \equiv (g^2 + g'^2)^{1/2}$.

Now let us assume that ϕ_1, $\mathbf{W}^{\bar a}$, and $\mathbf{A}$ are infinitesimal. Then, on discarding terms of cubic and higher order in these infinitesimal quantities, the energy integral can be written as

$$E = E_s + E_c + E_W , \quad (11)$$

where the semilocal string energy is [2]

$$E_s = \int d^2x [\tfrac{1}{4} (\partial_i Z_j - \partial_j Z_i)^2 + |d_j \phi|^2 + \lambda(\phi^\dagger\phi - \eta^2/2)^2] , \quad (12)$$

the contribution from the ϕ current and $\mathbf{W}^{\bar{a}}$ interaction is

$$E_c=\int d^2x\cos\theta_W J_j^{\bar{a}}W_j^{\bar{a}}\quad(\bar{a}=1,2)\,, \tag{13}$$

$$J_j^{\bar{a}}\equiv\tfrac{1}{2}ia[\phi^\dagger\tau^{\bar{a}}d_j\phi-(d_j\phi)^\dagger\tau^{\bar{a}}\phi]\,, \tag{14}$$

and the energy in the $\mathbf{W}^{\bar{a}}$ and $\mathbf{A}$ bosons is

$$E_W\equiv\int d^2x[\tfrac{1}{2}\gamma\mathbf{W}^1\times\mathbf{W}^2\cdot\nabla\times\mathbf{Z}+\tfrac{1}{4}|\nabla\times\mathbf{W}^1+\gamma\mathbf{W}^2\times\mathbf{Z}|^2+\tfrac{1}{4}|\nabla\times\mathbf{W}^2+\gamma\mathbf{Z}\times\mathbf{W}^1|^2+\tfrac{1}{4}g^2f^2(\mathbf{W}^{\bar{a}})^2+\tfrac{1}{4}(\nabla\times\mathbf{A})^2]\,, \tag{15}$$

where $\gamma\equiv g\cos\theta_W$. It may be remarked that the f and $\mathbf{Z}$ fields in (15) are the unperturbed fields of the semilocal string since we are only keeping up to quadratic terms in the infinitesimal quantities. Also, note that the current $J_j^{\bar{a}}$ is first order in the perturbation ϕ_1 because $(0,1)\tau^{\bar{a}}(0,1)^T=0$.

A comment about the semilocal energy is essential at this stage. In Refs. [2,6] the system that was considered was almost identical to the one given by the functional in Eq. (12). The only difference was in the matrix multiplying the gauge field inside the covariant derivative d_j. In the semilocal model, this matrix was the identity matrix whereas here it is the matrix T^1 defined in Eq. (10). This slight alteration will not make any difference in the existence of the vortex solution but might have some consequence for its stability.

The semilocal energy E_s has been shown [2] to be an extremum for the solution given in Eq. (7). To see this quickly, note that ϕ_1 only appears in quadratic and higher order in E_s and hence $\phi_1=0$ extremizes E_s. Once we set $\phi_1=0$, the energy functional E_s is identical to the energy functional for the Abelian-Higgs model where it is well known that vortex solutions exist [5]. These vortex solutions are precisely those given in Eq. (7).

The functional E_c is extremized by $\phi_1=0$ and $\mathbf{W}^{\bar{a}}=0$ since it is of quadratic order in these infinitesimal quantities. Similarly, E_W is extremized by the solution in Eq. (7) since this too is of quadratic order in the variations. This shows that, if we were to vary E with respect to the fields, we would find an extremal value for E when $W_\mu^1=0=W_\mu^2=A_\mu$ and ϕ and $\mathbf{Z}$ are given by Eq. (7). And with these values of the fields, the energy of the configuration is simply the energy of the Nielsen-Olesen solution found in Ref. [2].

This completes our proof that the semilocal vortex solution together with $W_\mu^1=0=W_\mu^2=A_\mu$ provides an extremum of the energy in the Weinberg-Salam model for all possible values of the coupling constants and all possible values of the Higgs boson mass. (In Ref. [3], we will demonstrate the solution directly from the equations of motion.) We now discuss the stability of the vortex to small perturbations.

We will first show that, for a range of parameters, the vortex solution described above is separately stable to small perturbations in the Higgs field and in the W and A fields. Then we will argue that a range of parameters exists such that the solution is stable to all perturbations.

Hindmarsh [6] has shown that the semilocal solution is stable to perturbations provided

$$\beta\equiv 8\lambda/a^2\le 1\,. \tag{16}$$

In our case, this translates into the existence of stable solutions that minimize E_{sl} (E_{sl} is E_s with $\theta_W=\pi/2$) provided

$$m_H\le m_Z\,, \tag{17}$$

where m_H and m_Z are the Higgs and Z boson masses, respectively.

The case with $m_H=m_Z$ leads to neutrally stable semilocal solutions [6] and so one can find perturbations such that $\delta E_{sl}=0$. However, when $m_H<m_Z$, the semilocal solution is stable to small perturbations and $\delta E_{sl}>0$ where we disregard the zero modes corresponding to global group transformations and spatial translations. This result has also been confirmed in recent simulations of semilocal string formation [7].

As pointed out above, the functional E_s in Eq. (12) is slightly different from the functional E_{sl} considered in Refs. [2,6] and so the results of the stability cannot be taken over directly from Ref. [6]. However, the arguments given by Hindmarsh indicate that even the functional E_s in Eq. (12) will be minimized by the vortex solution provided we have a small enough β. The argument runs as follows: By developing a nonvanishing ϕ_1, the potential term in E_s may be reduced. However, at the same time, the gradient terms increase. Since the potential term is proportional to β, a small enough β ensures that any change in the gradient term is more important than the change in the potential term. Hence, E_s will be a minimum when β is small enough. (In the case when $\theta_W=\pi/2$, this "small" value of β turned out to be 1.)

We now want to show that E_W in Eq. (15) is minimized by the vortex solution [8]. We have considered arbitrary perturbations of the fields and the only ones that can possibly lead to instabilities are found to take the following form:

$$\mathbf{W}^1=f_1(r)\cos(n\theta)\hat{\mathbf{e}}_r+\frac{h_1(r)}{r}\sin(n\theta)\hat{\mathbf{e}}_\theta\,, \tag{18}$$

$$\mathbf{W}^2=-f_2(r)\sin(n\theta)\hat{\mathbf{e}}_r+\frac{h_2(r)}{r}\cos(n\theta)\hat{\mathbf{e}}_\theta\,, \tag{19}$$

and $\mathbf{A}=0$.

The behavior of perturbations depends crucially on the profile of the vortex via the $\mathbf{Z}$ and f appearing in (15).

Therefore we need to know the equations satisfied by $\mathbf{Z}$ and f. If we write

$$\mathbf{Z} = -\frac{v(r)}{r}\hat{\mathbf{e}}_\theta , \tag{20}$$

the functions $f(r)$ and $v(r)$ satisfy the coupled equations

$$f'' + \frac{f'}{r} - \left(1 - \frac{\alpha}{2}v\right)^2 \frac{f}{r^2} - 2\lambda\left(f^2 - \frac{\eta^2}{2}\right)f = 0 , \tag{21}$$

$$v'' - \frac{v'}{r} + \alpha\left(1 - \frac{\alpha}{2}v\right)f^2 = 0 , \tag{22}$$

with $f \to \eta/\sqrt{2}$, $v \to 2/\alpha$ as $r \to \infty$ and $f \to 0$, $v \to 0$ as $r \to 0$. (Primes denote derivatives with respect to r.)

Now we insert Eqs. (18) and (19) into Eq. (15). For $n \neq 0$ this yields

$$E_W = \frac{\pi}{2}\int \frac{dr}{r}[-\gamma(f_1h_2 - f_2h_1)v' + \tfrac{1}{2}g^2f^2\{r^2(f_1^2+f_2^2) + h_1^2 + h_2^2\} + \tfrac{1}{2}\{nf_1 + \gamma v f_2 - h_1'\}^2 + \tfrac{1}{2}\{nf_2 + \gamma v f_1 + h_2'\}^2] . \tag{23}$$

For $n=0$, there is an additional factor of 2 and we should understand that all the θ dependence in Eqs. (18) and (19) has simply been dropped. In this case it is possible to combine the various terms in Eq. (23) to get

$$E_W = \frac{\pi}{2}\int \frac{dr}{r}\left\{\left[-\gamma v f_1 + \frac{v'}{v}h_2 - h_2'\right]^2 + \left[\gamma v f_2 + \frac{v'}{v}h_1 - h_1'\right]^2 + r\Delta(r)(h_1^2 + h_2^2) + (\nabla\times\mathbf{A})^2 + \tfrac{1}{2}g^2f^2[r^2(f_1^2+f_2^2) + h_1^2 + h_2^2]\right\} , \tag{24}$$

where

$$\Delta(r) \equiv \frac{1}{r}\left[\frac{v'}{rv} - \frac{v''}{v}\right] \propto -\frac{dB_Z}{dr} , \tag{25}$$

where B_Z is the Z magnetic-field strength of the string. Since the magnetic-field strength of the string is monotonically decreasing as we go away from the center of the string, $\Delta(r) \geq 0$. Therefore, all the terms under the integral in Eq. (24) are positive. This shows that the solution is stable against cylindrically symmetric perturbations.

We next turn to the $n \neq 0$ case. Now the calculation is considerably more tedious but the final result for E_W may be written as

$$E_W = E_f + \Delta[(h_1+h_2)/2;\gamma] + \Delta[(h_2-h_1)/2;-\gamma] , \tag{26}$$

where the integrand in E_f is a sum of squares of various combinations of the functions f_1 and f_2,

$$\Delta[\xi;\gamma] = \frac{\pi}{2}\int\frac{dr}{r}[A(r)\xi'^2 + S(r)\xi^2] \equiv \int dr\, \xi \mathcal{O} \xi , \tag{27}$$

$$A(r) = g^2r^2f^2/D(r) , \tag{28}$$

and

$$S(r) = g^2f^2 - \frac{(\gamma v')^2}{D} - \gamma r\frac{d}{dr}\left[\frac{v'(n+\gamma v)}{rD}\right] , \tag{29}$$

with

$$D(r) = (n+\gamma v)^2 + g^2r^2f^2 . \tag{30}$$

To determine if $\Delta[\xi;\gamma]$ is non-negative we must solve the eigenvalue problem associated with the operator $\mathcal{O}$ and find out if there are any negative eigenvalues. This will only be possible numerically. Here we will show that there exists a range of parameters for which both $A(r)$ and $S(r)$ are non-negative functions and hence $\Delta[\xi;\gamma] \geq 0$.

The function $A(r)$ is clearly non-negative. To check the positivity of the function $S(r)$, it is sufficient to check it at the origin since this is where the instability is most likely to occur. [As r gets large compared to the width of the string, $S \to g^2\eta^2/2 > 0$. Negative values of S can only occur in the region within the string, that is, in the region of small r. This may also be seen directly from Eq. (15) since $|\nabla\times\mathbf{Z}|$ is maximum at $r=0$ and decreases exponentially fast as we go away from the center of the string.] Then, using Eqs. (21) and (22) and the asymptotic values of the functions f and v near the origin,

$$f = (\eta/\sqrt{2})f_0 r + \cdots , \tag{31}$$

$$v(r) = \frac{\alpha}{2}\left[\frac{v_0}{2!}r^2 + \frac{v_1}{4!}r^4 + \cdots\right] , \tag{32}$$

where $v_0, f_0 > 0$, we find

$$S(r) = \tfrac{1}{2}g^2\eta^2f_0^2r^2(1 \pm 1/n) \quad (r \to 0) , \tag{33}$$

where the two signs correspond to the last two terms in Eq. (26). Clearly $S(r)$ is positive near the origin for $n \geq 2$. For the case with $n=1$, we have to find $S(r)$ to order r^4:

$$S(n=1) = \frac{v_0f_0^2g^2}{2}(1 - 18\cos^2\theta_W)r^4 \quad (r \to 0) . \tag{34}$$

Therefore if $\sin^2\theta_W > \frac{17}{18} \approx 0.94$ then $S(r) \geq 0$ and the vortex solution is stable to perturbations in the W fields.

Note that the condition $S(r) \geq 0$ is a sufficient condition for the positivity of E_W. So we expect that E_W will also be positive for values of $\sin^2\theta_W$ somewhat less than 0.94. Also note that for large $\sin^2\theta_W$, $E_W=0$ if and only if $\mathbf{W}^{\bar{a}}=0$.

This shows that, if we consider the change in the energy of the vortex solution when the fields are varied, we will find $\delta E_s \geq 0$ and $\delta E_W \geq 0$ provided β is small and $\sin^2\theta_W$ is large. This shows that the vortex solution is stable to perturbations in the Higgs and Z fields and, separately, to perturbations in the W fields (because in these cases $E_c=0$). We now wish to discuss the case when *both* the Higgs and W fields are perturbed simultaneously and $E_c \neq 0$.

Let us consider the energy functional E in the limit that $\sin\theta_W \to 1$. In this limit, we find

$$E \to E_{\rm sl} + \epsilon \int d^2x\, J_j^{\bar{a}} W_j^{\bar{a}} + E_{W\,\rm free} + O(\epsilon^2)\,, \tag{35}$$

where $\epsilon \equiv \pi/2 - \theta_W \to 0$ and $E_{W\,\rm free}$ is E_W [Eq. (15)] with γ set equal to zero. Now, if we consider perturbations in ϕ_1 and $W_j^{\bar{a}}$, the change in $E_{\rm sl}$ and $E_{W\,\rm free}$ is given by

$$\delta E_{\rm sl} = O(\epsilon^0) > 0, \quad \delta E_{W\,\rm free} = O(\epsilon^0) > 0\,, \tag{36}$$

for $\beta < 1$ while the change in the second term in (35) is clearly $O(\epsilon^1)$. Therefore, for small enough ϵ, we have $\delta E \geq 0$ and the vortex solution is stable.

"Zero modes," such as global SU(2) transformations on ϕ in $E_{\rm sl}$ and also $\mathbf{W}^{\bar{a}} = \nabla F^{\bar{a}}$ in $E_{W\,\rm free}$, require special attention since, for these, $\delta E_{\rm sl}=0$ and $\delta E_{W\,\rm free}=0$ and it would seem that δE can be made negative via the second term in (35). The best way to resolve this issue is to realize that, for example, $\mathbf{W}^{\bar{a}} = \nabla F^{\bar{a}}$ can be transformed into $\mathbf{W}^{\bar{a}}=0$ (to order ϵ^2) by an infinitesimal gauge transformation. This eliminates the second term in (35) and we are left only with positive contributions to E [Eq. (36)]. Similar considerations apply to the global SU(2) transformation of ϕ in $E_{\rm sl}$.

How can we understand the stability of the vortex solution? One way is to think of the Weinberg-Salam model as a two-parameter family of models. One of the parameters is β and the other is θ_W. In this two-parameter space, the line segment $\theta_W = \pi/2$, $0 < \beta \leq 1$, gives stable vortex solutions that have been called semilocal strings. Further, the semilocal string solution gets more stable as the parameter β is decreased from unity. (The end point, $\beta = 1$, is where the strings are neutrally stable and is the border between stable and unstable semilocal strings.) Hence, by continuity, the solutions must be stable for parameters in the neighborhood of the line $\theta_W = \pi/2$, $0 < \beta < 1$.

To summarize, we have shown (i) the existence of vortex solutions in the Weinberg-Salam model for arbitrary values of the parameters, and (ii) the existence of values of the parameters such that the solution is stable to small perturbations.

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