A Nonsingular Universe

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The Einstein theory of gravitation breaks down at very high curvatures. We propose a modification of the theory at Planck curvatures for which all isotropic cosmological solutions (even including matter) are nonsingular. All solutions asymptotically approach de Sitter space, a solution with limiting curvature.

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As was shown by Penrose and Hawking [1], under certain reasonable conditions the space-times in general relativity are singular in the sense that they are geodesically incomplete. The two most important examples of singularities are the big bang singularity of cosmological models of the Universe and the singularity at the center of a black hole.

The crucial assumptions which enter the proofs of the singularity theorems are, first, the universal applicability of the Einstein equations and, second, the weak energy dominance condition [1]. These theorems do not tell us anything about the nature of the singularities. However, based on some well-known black hole and cosmological solutions, it is rather natural to assume that some of the curvature and energy-momentum tensor invariants become infinite at the singularity (for singularities reached on a timelike curve in a globally hyperbolic space-time it can in fact be proven that the Riemann tensor becomes infinite [2]). But, if this is the case, then the applicability of the main assumptions used to prove the singularity theorems becomes questionable. Most importantly, the Einstein theory breaks down at high curvatures, as can be seen, for example, from perturbative calculations of the effective action [3]. Moreover, if string theory is the correct theory, then the Einstein action is just an effective theory valid at low energies.

It thus seems clear that the Einstein equations must be modified at high curvatures. We may then hope that the singularity problem which has sometimes been viewed as a "crisis in physics" [4] can be solved in some new theory of gravitation. The most plausible candidates for such a new theory are quantum gravity and string theory (for a first attempt at a string cosmology without singularities see, e.g., Ref. [5]). Unfortunately, at the present time it is not known what these theories mean beyond their perturbative content. It is unlikely that the fundamental problem of singularities can be solved using perturbation theory. Thus, we must guess the form of a theory without singularities based on some new physical principles.

A natural hypothesis is to assume the existence of a length l_f, fundamental in the sense that no length $l < l_f$ can be measured and all physical observables must be smeared out on a scale l_f. There are well-known examples, such as special relativity (where $v \leq c$) and quantum mechanics (where $\Delta p \Delta x \geq \hbar/2$), in which the assumption that some physical quantity is bounded leads to a new theory.

Since the third fundamental constant, the gravitational constant G, has not been exploited in the above-mentioned context, it is rather natural to conjecture that the principle $l \geq l_f \sim (G\hbar c^{-3})^{1/2}$ could lead to a new theory. Since the gravitational constant is of the same order of magnitude as the string theory constant, we may hope that the limiting length is already realized in nonperturbative string theory (for an indication of the existence of a limiting length in string theory see, e.g., Ref. [6]).

If there is a fundamental length $l_f \sim l_{\rm Pl}$, then it follows from dimensional arguments that all curvature invariants (not just the fourteen algebraic invariants) must be bounded [7]:

$$|R| < 1/l_{\rm Pl}^2, \quad |R_{\mu\nu}R^{\mu\nu}| < 1/l_{\rm Pl}^4, \quad |R_{\mu\nu;\delta}R^{\mu\nu;\delta}| < 1/l_{\rm Pl}^6, \ldots \tag{1}$$

In this case, there would be no singularities.

The purpose of this Letter is to show how to modify the Einstein action in order to satisfy (1) and obtain a theory in which all solutions for an isotropic universe are nonsingular. As the reader will see, the considerations can be generalized to an arbitrary geometry. We believe that the theory which we will construct is reasonable and could arise as an effective theory in some more fundamental models such as string theory or quantum gravity.

We shall consider a classical local theory since it is the simplest model. However, if the problem of singularities can be solved at this level, we may hope that the problem can also be solved using similar techniques in quantum generalizations of the theory, and also in models including all possible nonlocalities.

Limiting curvature hypothesis.— Even if some of the low-order curvature invariants (1) are bounded, this does not imply that all higher-order invariants vanish. For example, we could have $|R_{\mu\nu}R^{\mu\nu}| < 1/l_{\rm Pl}^4$ but nevertheless $|R_{\mu\nu;\delta}R^{\mu\nu;\delta}| \to \infty$. In this case there would also be a singularity. In general, it is a formidable task (if it is possible at all) to construct a theory in which an infinite set of invariants is bounded in an explicit manner. However, we can reduce the problem by introducing the limiting curvature hypothesis (LCH) [7,8] which states that (i) a finite number of invariants are bounded (e.g.,

$|R| \leq 1/l_{\rm Pl}^2$ and $|R_{\mu\nu}R^{\mu\nu}| \leq 1/l_{\rm Pl}^4$), and (ii) when these invariants take on their limiting values, any solution of the field equations reduces to a definite nonsingular solution (e.g., de Sitter space), for which all invariants are automatically bounded.

If the limiting space is de Sitter, then $C^2=0$ (where C is the Weyl tensor). Hence, when applied to a cosmological model the LCH encompasses Penrose's "vanishing Weyl tensor conjecture" for the initial state of the universe [9]. For an isotropic universe, the LCH implies that there are no initial and final singularities. A de Sitter phase is the initial and final stage of the evolution. (For speculations on how it is possible to have no contradictions with the second law of thermodynamics see Ref. [10].) When applied to black holes, the LCH implies that instead of a singularity there is a de Sitter space inside the Schwarzschild horizon, which can in turn spawn a new Friedman "baby" universe [8].

The purpose of this Letter is to describe a theory which encompasses the LCH in a natural manner. In order to achieve this goal, we start with a modification of the usual Einstein action for the gravitational field and matter. We assume that the true "effective" or fundamental action for the gravitational field includes higher-order curvature invariants,

$$S=\int F(R,R_{\mu\nu}R^{\mu\nu},C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma},\dots)\sqrt{-g}\,d^4x\,, \quad (2)$$

and at low energies reduces to the Einstein action. In this simple model, we do not consider nonlocal terms and possible modifications of the matter part of the action (see Ref. [11] for an attempt to modify only the matter sector). However, since we are able to construct a realization of the LCH, we can on that basis justify neglecting nonlocal terms in this model.

Note that modified actions of the form (2) arise in many contexts, e.g., when considering the effective action for the gravitational modes of string theory [12], when quantizing matter fields in a nontrivial background metric [13], or when attempting perturbative quantum gravity calculations [3].

The gravitational action (2) leads to a higher derivative theory with many additional solutions. In general, the singularity problem for this kind of action will be worse than for Einstein gravity, since even without matter these theories admit a lot of singular solutions. However, we shall construct a theory for which, even in the presence of matter, the LCH is realized and hence all solutions are nonsingular.

Theory.—To understand why it is possible to construct a theory in which the LCH is realized, it is useful to consider well-known theories with bounded physical quantities such as special relativity and the Born-Infeld theory [14].

To express the bounds on physical quantities in an explicit manner, it is convenient to rewrite the actions of these theories by introducing a nondynamical scalar field φ [15]. For example, the action for a point particle in special relativity may be rewritten as

$$S=m\int[\tfrac{1}{2}\dot{x}^2+\varphi\dot{x}^2-V(\varphi)]dt\,, \quad (3)$$

with potential $V(\varphi)=2\varphi^2/(1+2\varphi)$. Varying with respect to φ yields the constraint

$$\dot{x}^2=\frac{\partial V}{\partial\varphi}=1-\frac{1}{(1+2\varphi)^2} \quad (4)$$

from which it is obvious that $\dot{x}^2$ is bounded ($\dot{x}^2\leq 1$). In fact, solving the constraint equation for φ in terms of $\dot{x}^2$ and substituting the result into (3) we obtain (up to a constant) the standard form of the action for special relativity.

In the following, we shall generalize the above procedure to gravity. In order to impose bounds on a finite number of curvature invariants $I_1,\dots,I_n$ in an explicit manner, we rewrite the action (2) in the form

$$S=-\frac{1}{16\pi G}\int\left(R+\sum_{i=1}^{n}\varphi_i f_i(I_i)+V(\varphi_1,\dots,\varphi_n)\right)\times\sqrt{-g}\,d^4x\,, \quad (5)$$

where φ_i are additional nondynamical scalar fields and f_i are given functions. In fact, any nondegenerate action (2) can be rewritten in the above form [10], provided that the potential V is chosen appropriately. We will choose a potential $V(\varphi_1,\dots,\varphi_n)$ which ensures that the bounds on $I_1,\dots,I_n$ are automatically imposed.

To concretize the consideration, we will demonstrate how to construct the action for a theory which encompasses the LCH in the case of a spatially flat, isotropic universe.

For an isotropic universe, the Weyl tensor C vanishes. To realize the LCH it proves sufficient in this case to impose bounds on the two simplest curvature invariants $I_1=R$ and $I_2=4R_{\mu\nu}R^{\mu\nu}-R^2$. Note that $I_2\geq 0$ for any spherically symmetric metric and that (given that C vanishes) $I_2=0$ only for Minkowski and de Sitter universes. Thus, to ensure that the asymptotic solutions approach de Sitter space it is sufficient to have $I_2\to 0$ for all solutions as the limiting curvature is approached.

To obtain bounded curvature we can impose constraints on some combination of I_1 and I_2 similar to what was done for special relativity. Thus, for an appropriate choice of the potential $V(\varphi_1,\varphi_2)$, a theory with action

$$S=-\frac{1}{16\pi G}\int[R+\varphi_1 f_1(I_1)+\varphi_2 f_2(I_2)+V(\varphi_1,\varphi_2)]\sqrt{-g}\,d^4x \quad (6)$$

can satisfy the LCH.

The model.—The simplest of the actions of type (6) for which the LCH can be realized takes the form (see Ref. [10] for details)

$$S=-\frac{1}{16\pi G}\int\left[(1+\varphi_1)R-\left(\varphi_2+\frac{6}{\sqrt{12}}\varphi_1\right)(4R_{\mu\nu}R^{\mu\nu}-R^2)^{1/2}+V_1(\varphi_1)+V_2(\varphi_2)\right]\sqrt{-g}\,d^4x\,. \tag{7}$$

In this theory, Minkowski space is stable against small inhomogeneous fluctuations and there are no tachyonic modes. Note that there is only little freedom in the choice of this action. Even small deviations from (7) yield theories with many singular solutions.

It is worth pointing out that if we set $\varphi_1=0$ and $V_1=0$ in the action (7), i.e., we consider a model with only a single scalar field φ_2, then also in the resulting theory all isotropic cosmological solutions will be nonsingular, even when matter is included. However, there is no general bound on the curvature valid for all trajectories [10]. The second scalar field φ_1 and the corresponding constraint are required in order to bound the curvature invariant for these nonsingular solutions.

In this Letter we shall consider a contracting spatially flat isotropic universe and show that for all initial conditions and for general matter content the final state is de Sitter space. Hence, there will be no final "big crunch." In a similar way, the action (7) also solves the initial singularity problem.

Of course, to obtain a contracting universe we should consider a closed universe. However, for scale factors much smaller than the maximal one $a_{\max}$ (for a closed matter-dominated universe) and much larger than the minimum one $a_{\min}$ (for a closed de Sitter universe), the flat space cosmological model is a good approximation. The only change in a closed universe at large curvatures is—as will be shown in a detailed paper [10]—that de Sitter space leads to a "bounce" and a reexpansion of the universe.

For a contracting universe, the variational equations which follow from the action (7) take the form [10]

$$H^2=\frac{1}{12}V_1',\quad \dot H=-\frac{1}{\sqrt{12}}V_2',$$
$$3(1-2\varphi_1)H^2-\frac{1}{2}(V_1+V_2)=\frac{6}{\sqrt{12}}H(\dot\varphi_2+3H\varphi_2)\,, \tag{8}$$

where $H=\dot a/a$ (a being the scale factor), $V_i'=\partial V_i/\partial\varphi_i$ for $i=1,2$ and an overdot denotes derivative with respect to physical time.

The potentials $V_1(\varphi_1)$ and $V_2(\varphi_2)$ must be chosen such that for $|\varphi_1|,|\varphi_2|\ll 1$ the leading terms in the action give back the Einstein theory. A sufficient condition is $V_i(\varphi_i)\sim\varphi_i^2$ for $|\varphi_i|\ll 1$. The asymptotic behavior of the potentials at large $|\varphi_i|$ can be obtained by demanding that $I_1+(6/\sqrt{12})I_2^{1/2}$ (which is proportional to H^2) is bounded (limiting curvature) and that $I_2\sim\dot H^2$ tends to zero at large $|\varphi|$ (de Sitter universe). This requires

$$V_1'\to\text{const at }|\varphi_1|\to\infty\text{ and }V_2'\to 0\text{ as }|\varphi_2|\to\infty\,. \tag{9}$$

The solution of Eqs. (8) can be reduced to analyzing the following ordinary first-order differential equation:

$$\frac{d\varphi_2}{d\varphi_1}=\frac{V_1''}{V_1'V_2'}\left[-\frac{1}{4}(1-2\varphi_1)V_1'+\frac{1}{2}(V_1+V_2)+\frac{3}{2\sqrt{12}}V_1'\varphi_2\right], \tag{10}$$

which allows a complete investigation using the phase diagram method.

For the specific choice of the potential asymptotics

$$V_1(\varphi_1)\propto\varphi_1-\ln(\varphi_1)\,,$$
$$V_2(\varphi_2)=\text{const}-\frac{1}{\varphi_2}+\cdots\quad\text{as }|\varphi_1|,|\varphi_2|\gg 1\,, \tag{11}$$

which satisfies the conditions (9), the trajectories are sketched in the φ_1-φ_2 plane in Fig. 1, with arrows indicating time evolution. Qualitatively, there are four different regions delineated by separatrices. In all regions the solutions are nonsingular, and in the asymptotic regions ($|\varphi_1|+|\varphi_2|\gg 1$) they correspond to de Sitter universes. A detailed investigation of the phase diagram will be presented in Ref. [10], where it will also be shown that the qualitative form of the phase diagram is the same for any potentials V_1 and V_2 which satisfy (9).

An interesting question is: What will happen if matter is incorporated into the model? We have investigated [10] the behavior of spatially flat and closed universes

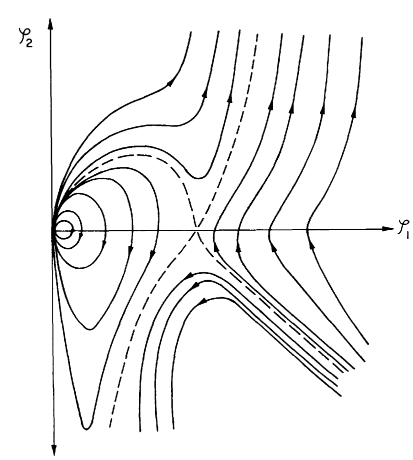

FIG. 1. Phase diagram for the solutions of (10), arrows pointing in the direction of increasing time. As can be shown using (8), the asymptotic solutions are de Sitter solutions.

containing dustlike matter and radiation using numerical methods. We showed that also in these cases all solutions are nonsingular and asymptotically approach de Sitter space at high curvatures. The matter only influences the rate of the asymptotic approach of the solution to de Sitter space. Matter terms are subdominant in the asymptotic region. In this sense, the theory under consideration incorporates the property of asymptotic freedom at high curvatures.

Note that the effective "cosmological constant" which is responsible for the de Sitter phase has a dynamical origin and is not put into the theory by hand.

In conclusion, in this Letter we have presented a theory of gravity whose action contains terms which modify the Einstein equations at large curvatures and for which all isotropic cosmological solutions (not only special solutions as in some other models [16]) are nonsingular, regardless of the matter content of the universe. Asymptotically, the solutions approach a de Sitter universe with a limiting curvature. Hence, the space-time is geodesically complete.

Our analysis can be considered as a search for an effective field theory of gravity close to the Planck scale (but still in the domain where classical space-time notions apply), under the assumption that this theory is able to solve the singularity problem.

From our considerations, it is clear how to generalize the above theory to situations when the Weyl curvature tensor does not vanish in order to obtain only nonsingular solutions. An example arises in the case of black holes. These extensions will be discussed in a separate paper [17].

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