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Zeta Function for the Lyapunov Exponent of a Product of Random Matrices

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A cycle expansion for the Lyapunov exponent of a product of random matrices is derived. The formula is nonperturbative and numerically effective, which allows the Lyapunov exponent to be computed to high accuracy. In particular, the free energy and the heat capacity are computed for the one-dimensional Ising model with quenched disorder. The formula is derived by using a Bernoulli dynamical system to mimic the randomness.

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The product of random matrices often appears in the study of disordered materials and of dynamical systems. The physical quantities of these systems are related to the rate of growth of the random product—the Lyapunov exponent. For example, in the study of an Ising model with quenched randomness the Lyapunov exponent is proportional to the free energy per particle; in the Schrödinger equation with a random potential, the Lyapunov exponent is proportional to the localization length of the wave function; and in the motion of a classical particle, the Lyapunov exponent indicates the degree of sensitivity to initial conditions (chaos). Since Dyson [1] studied a system of harmonic oscillators with random couplings, many problems have been reduced to the study of a Lyapunov exponent. In these problems the product of random matrices appears when a discrete version of a differential operator is considered, or when the problem is solved on a lattice. Further applications of Lyapunov exponents, and related derivations, are reviewed in the paper by Alexander *et al.* [2] and the book by Crisanti, Paladin, and Vulpiani [3].

There are few analytic results for the Lyapunov exponent of a product of random matrices, and very little has been determined about related systems without resorting to Monte Carlo simulations. A theorem of Oseledec [4] states that the norm of the random product grows exponentially with the number of multiplied terms at a rate given by the Lyapunov exponent, but the theorem does not provide a method for determining the exponent. The two known methods for calculating the Lyapunov exponent, weak disorder expansions [5–7] and microcanonical approximations [8], have limitations. The weak disorder expansion imposes conditions on the eigenvalues of the matrices and is difficult to carry out to high orders; and the microcanonical method, while general, does not provide a systematic expansion and is difficult to apply to large matrices. In this Letter a formula for computing the Lyapunov exponent will be derived. It is simple to evaluate and is nonperturbative in character, with the first few terms providing a good numerical approximation. In particular, it gives all thermodynamic quantities for the one-dimensional Ising model with a discrete valued random magnetic field when the disorder averaging is done over the free energy (this is the more difficult quenched disorder case).

The formula is obtained by viewing the random product as a statistical mechanical system which is solved using the cycle expansion [9] of its thermodynamical zeta function [10]. Cycle expansions have been very successful in obtaining nonperturbative expansions of chaotic dynamical systems [11,12], of generalized Ising systems [13,14], and of scattering problems in quantum mechanics [15].

We will consider the product

$$G^{(n)} = \prod_{0<k\leq n} T_k \tag{1}$$

of matrices T_k chosen at random from a discrete set. This includes many cases of interest and can be used to approximate a continuous distribution. The maximal Lyapunov exponent γ can be expressed [16] as the rate of exponential growth of the norm of the product $G^{(n)}$ with the number n of matrices multiplied:

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\langle \ln ||G^{(n)}|| \rangle . \quad (2)$$

The average is over all possible realizations of the product, each product taken with the appropriate probability. The theorem of Oseledec [4] guarantees that the limit exists for almost every realization. The definition of the Lyapunov exponent appears to depend on the matrix norm chosen, but it can be shown [16] that its value remains unchanged as long as equivalent norms are used. For the finite-dimensional vector space of $n\times n$ matrices all norms are equivalent, making the Lyapunov exponent independent of the norm chosen.

Because it is very difficult to handle the logarithm inside the average we will substitute a power and a derivative for the logarithm, and write the Lyapunov exponent as

$$\gamma = \lim_{n\to\infty} \partial_\alpha \frac{1}{n}\langle ||G^{(n)}||^\alpha \rangle|_{\alpha=0} . \quad (3)$$

To determine the averages, introduce the generating function

$$\zeta(z,\alpha) = \exp\left[\sum_{n\geq 1} \frac{z^n}{n}\langle ||G^{(n)}||^\alpha \rangle\right] , \quad (4)$$

which is the Ruelle zeta function [13] for a statistical mechanical system with $\langle ||G^{(n)}||^\alpha\rangle$ as the free energy. The zero $\hat{z}(\alpha)$ of $1/\zeta(z,\alpha)$ gives the exponential growth of $\langle ||G^{(n)}||^\alpha\rangle$ (see Ref. [13], Theorem 5.29]), and by using the special values $\hat{z}(0)=1$ and $\partial_z\zeta^{-1}(1,0)=-1$, the Lyapunov exponent can be reexpressed as

$$\gamma = -\partial_\alpha \ln \hat{z}(\alpha=0) = -\partial_\alpha \zeta^{-1}(1,0) . \quad (5)$$

The expression in terms of the derivative of the zeta function is of no advantage unless it can be computed in an efficient manner. If the terms of the zeta function satisfy certain combinatorial properties the inverse zeta function can be written as a cycle expansion [9], which is rapidly convergent and offers a practical scheme for evaluating the Lyapunov exponent. The average $z^n\langle ||G^{(n)}||\rangle$ is the sum of terms of the form

$$t_G = z^{n(G)}\mathrm{Prob}(G)||G||^\alpha , \quad (6)$$

with $n(G)$ being the number of matrices in the product G. If the weights are cyclic, as in $t_{AAB}=t_{ABA}=t_{BAA}$, and multiplicative, as in $t_{ABAB}=(t_{AB})^2$, then the inverse zeta function can be expanded into a cycle expansion [11,14]

$$\zeta^{-1}(z,\alpha) = \prod_{G\in P}(1-t_G) , \quad (7)$$

with the product being over the set P of all possible prime products. A product of matrices is prime if it is not the repeat of a smaller length product. Two products are equivalent for the expansion if they differ by a cyclic rotation. For example, if AB is in the set P, then BA does not need to be in the set as it is equivalent to AB by cyclic rotation. Also $ABAB$ and $BABA$ need not be in P as they are repeats of AB or of its cyclic rotation.

To continue the derivation we use the independence of the Lyapunov exponent on the norm, and choose the most convenient norm for the cycle expansion. If we choose the norm absolute value of the largest eigenvalue (or eigenvalues), and write it in the peculiar form

$$||G||^\alpha = \lim_{n\to\infty} |\mathrm{tr}G^n|^{\alpha/n} , \quad (8)$$

then it is simple to verify that the weights t_G are multiplicative and cyclic. They are multiplicative because

$$||G^p||^\alpha = \lim_{n\to\infty} |\mathrm{tr}G^{pn}|^{\alpha/n} = \left(\lim_{pn\to\infty} |\mathrm{tr}G^{pn}|^{1/pn}\right)^{\alpha p} = ||G||^{\alpha p} , \quad (9)$$

and they are cyclic because the trace is cyclic. The Prob($\cdot$) part of the weight is multiplicative and cyclic, as is the product of numbers. The cycle expansion for $\langle ||G^{(n)}||^\alpha\rangle$ is then obtained by expanding the infinite product

$$\zeta^{-1}(z,\alpha) = \prod_{G\in P}[1-z^{n(G)}\mathrm{Prob}(G)||G||^\alpha] \quad (10)$$

into a power series in z. This is essential when computing the zeros of ζ^{-1}. The power series in z converges as long as the matrices are hyperbolic (not all eigenvalues are equal to 1 in modulus). The cycle expansion (10) can be used to compute the Lyapunov exponent in Eq. (5). In the case that there are two matrices forming the random product, A with probability p and B with probability $q=1-p$, the first few terms of the expansion of (5) are

$$-\gamma = p\ln||A|| + q\ln||B|| + pq(\ln||AB|| - \ln||A|| - \ln||B||) + ppq(\ln||AAB|| - \ln||AB|| - \ln||A||) + pqq(\ln||ABB|| - \ln||AB|| - \ln||B||) + \cdots . \quad (11)$$

General expressions and a more detailed derivation can be found in Ref. [17].

One test of the expansion for the Lyapunov exponent is a product of random matrices that appears in the study of the one-dimensional Ising model with coupling constant J and a random magnetic field assuming the values $\pm h$. The two matrices, chosen with equal probability and after factoring out a common term, are of the form

$$\begin{pmatrix} 1 & ab^k \\ a & b^k \end{pmatrix} , \quad (12)$$

where k is $+1$ or -1, a is $\exp(-2J)$, and b is $\exp(-2h)$. In this case both matrices have eigenvalues

TABLE I. The Lyapunov exponent and its second derivative (proportional to the heat capacity) for the transfer matrices [see Eq. (12)] when $j=0.3$ and $h=1.4$. The Monte Carlo and zeta function values are quoted so that errors are limited to the last digit.

Method	Lyapunov	Heat capacity
Weak disorder	1.5	...
Microcanonical	1.20	...
Monte Carlo	1.1773	0.4
Zeta ($n=15$)	1.177273613342 68	0.3664

that are real and different from 1, and are therefore hyperbolic. The Lyapunov exponent for these matrices can also be computed by Monte Carlo simulations, weak disorder expansions, and microcanonical approximations. Table I has the results for these methods. The Monte Carlo calculation corresponds to 128 realizations of a product of 10^6 matrices; the weak disorder expansion is carried to fifth order; there is no error estimate for the microcanonical method, except for a rigorous upper bound; and the value for the zeta function method was computed by including cycles up to length fifteen. The zeta function expansion takes 17 s on a Sparcstation 1 computer, whereas if the Monte Carlo simulation had to be carried out to the precision obtained in the cycle expansion, it would require several hundred years of Sparcstation 1 time. The weak disorder expansion could in principal match the accuracy of the zeta function expansion if the terms of higher order were known. Also in Table I the second derivative of the Lyapunov exponent (proportional to the heat capacity at $\beta=1$) is computed by numerical differentiation. Notice that, except for the zeta function, all analytic methods fail to provide a value for the second derivative and that even though Monte Carlo simulation does estimate the derivative with one digit, a better estimate would require a prohibitive amount of computer time.

To study the convergence of the zeta function method one can plot the rate at which digits are gained in the value of the Lyapunov exponent as longer cycles are included in the expansion. The better a system is understood, the better the nature of the convergence. In Fig. 1 the number of correct digits as a function of the largest cycle considered is plotted. If γ_n is the approximation to the Lyapunov exponent when cycles up to length n are included, then the number of digits is defined as $d(n) = \log_{10}(\gamma_{n-1} - \gamma_n)$. The straight line indicates that the convergence is exponentially fast in the length of the product.

To further illustrate this method, Fig. 2 has a plot of the Lyapunov exponent for the Ising model pair of matrices chosen with equal probability and in units where $J=h=\beta$. In the plot all points can be computed to machine precision, and the convergence rate is similar to that of Fig. 1. Thus the method is not limited to small

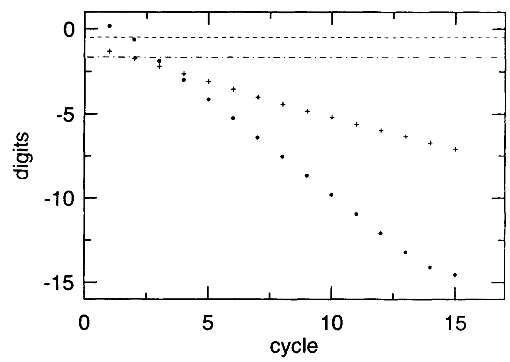

FIG. 1. Number of digits that remain constant as longer cycles are included in the expansion of the Lyapunov exponent. The solid circles are for the random Ising model and the crosses are for the degenerate 3×3 matrices. Also indicated in the plot are the accuracy of the weak disorder expansion (dot-dashed line) and the microcanonical approximation (dashed line).

values of the inverse temperature β as the weak disorder expansion, and can be used to obtain thermodynamic quantities at any temperature.

The weak disorder expansion cannot be applied when there are repeated eigenvalues, so to illustrate the zeta function method the Lyapunov exponent has been computed for a pair of matrices with degenerate eigenvalues. The random products are formed from a pair of 3×3 matrices, which have the same eigenvalues, do not commute, and are not related by a similarity transformation. The eigenvalues are 2, 2, and 1. The exponential convergence of the method is not affected by the largest eigenvalue being degenerate, nor by the presence of an indifferent eigenvalue (the one of value 1). In Fig. 1 the crosses are the plot of the number of nonchanging digits of the Lyapunov exponent as a function of the cycle length.

The cycle expansion developed in this Letter for the Lyapunov exponent is an efficient computational tool. It can be applied to a wide variety of matrix products

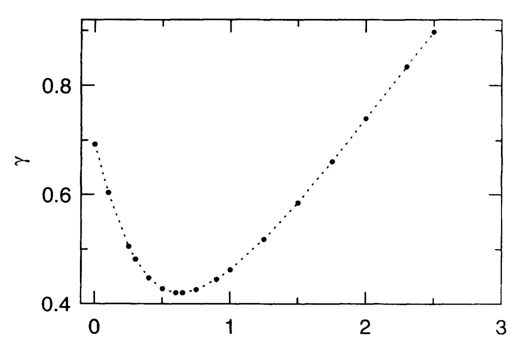

FIG. 2. Lyapunov exponent (free energy) for the pair of matrices [Eq. (12)] that describe the Ising model with quenched randomness as a function of the inverse temperature β. The dotted line is an interpolation of the computed points.

without the limitations of other methods, and excludes only the matrices where all the eigenvalues are one in modulus. The method has been successfully applied to Ising models with a random magnetic field on a strip [17], and also in reproducing the branch point at zero temperature predicted by Derrida and Hilhorst [18] in the same model.

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