

Multiscaling and the Classification of Continuous Phase Transitions

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Multiscaling of the free energy is obtained by generalizing the classification of phase transitions proposed by Ehrenfest. The free energy is found to obey a new generalized scaling form which contains as special cases standard and multiscaling forms. The results are obtained by analytic continuation from the classification scheme of Ehrenfest.

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An increasing number of recent publications report a novel type of scaling behavior which has been termed multiscaling [1-11]. Particular examples were found in spinodal decomposition [3-5], turbulence [6,7], sandpile models [2], diffusion-limited aggregation [1,10], and frequency-dependent dielectric data for supercooled liquids [11]. In view of the importance of scaling concepts in many branches of physics these findings are of great general interest.

Multiscaling will be defined here as a scaling form in which the critical exponents may become scale dependent. This terminology was introduced by Coniglio and Zanetti [3] to express the similarity between multiscaling and multifractals. Indeed it was pointed out recently [8] that multiscaling follows generally from multifractality whenever a lower cutoff is introduced in the calculations of correlation functions. It should be noted that multiscaling is a generalization of standard scaling and contains the latter as the special case in which the exponent function is a constant.

Despite its generality and early appearance [12] multiscaling forms have not found much attention in statistical physics. It should therefore be interesting that different forms of multiscaling appear naturally already at the level of classical thermodynamics. The purpose of this paper is twofold: to show (1) that the Ehrenfest classification of phase transitions applies to continuous transitions in exactly the same way as to first-order transitions and (2) that this observation gives rise to different forms of multiscaling.

General phase transitions were classified by Ehrenfest [13] according to their order. Ehrenfest defined a p th-

order transition as one in which the j th-order derivatives of a thermodynamic potential with $j \leq p-1$ are continuous while the p th derivative shows a jump discontinuity. To be specific consider a magnetic system whose free energy $F(t, h)$ is written as a function of reduced temperature $t = (T - T_c)/T_c$ and the ordering field h . The origin ($t=0, h=0$) is assumed to belong to a critical manifold, and represents the critical point of interest.

Let $\mathcal{C}: \mathbb{R} \rightarrow \mathbb{R}^2$, $s \rightarrow (t(s), h(s))$ be an arbitrary smooth curve through the critical point such that $t(0)=0$ and $h(0)=0$. The phase transition can now be classified by applying Ehrenfest's idea to the singular part of the free energy restricted to the curve \mathcal{C} . Define the limits

$$A^\pm(\mathcal{C}) = \lim_{s \rightarrow 0^\pm} \frac{d^p F_{\text{sing}}(t(s), h(s))}{ds^p} \quad (1)$$

for the p th derivative of the singular part of the free energy restricted to the curve \mathcal{C} . The order of the phase transition along the curve \mathcal{C} is defined by Ehrenfest as the smallest integer $p(\mathcal{C}) \geq 1$ such that $A^+(\mathcal{C}) \neq A^-(\mathcal{C})$. Many authors have discussed the inadequacy of Ehrenfest's classification [14-16]. As a consequence, it has become customary to distinguish only between first-order and continuous transitions. However, it will be shown here that Ehrenfest's classification remains generally adequate and that analytic continuation in p allows us to classify continuous transitions precisely according to their order.

The key step towards a quantitative distinction between continuous phase transitions of different order is to rewrite (1) in terms of a finite difference quotient. Rewriting (1) for the singular part of the free energy then gives

$$A^\pm(\mathcal{C}) = \lim_{s \rightarrow 0^\pm} \lim_{N \rightarrow \infty} \left(\frac{|s|}{N} \right)^{-pN-1} \sum_{j=0}^{N-1} (-1)^j \binom{p}{j} F \left[t \left(s \mp \frac{js}{N} \right), h \left(s \mp \frac{js}{N} \right) \right], \quad (2)$$

where p is an arbitrary integer variable and the subscript of F has been suppressed to shorten notation. This reformulation of Eq. (1) can be analytically continued in p to give

$$A^\pm(\mathcal{C}) = \lim_{s \rightarrow 0^\pm} \lim_{N \rightarrow \infty} \Gamma(-p)^{-1} \left(\frac{|s|}{N} \right)^{-pN-1} \sum_{j=0}^{N-1} \frac{\Gamma(j-p)}{\Gamma(j+1)} F \left[t \left(s \mp \frac{js}{N} \right), h \left(s \mp \frac{js}{N} \right) \right], \quad (3)$$

where now $p \in \mathbb{R}$. Using the analyticity away from the critical point it can then be shown that

$$A^\pm(\mathcal{C}) = \lim_{s \rightarrow 0^\pm} \Gamma(-p)^{-1} |s|^{-p} \sum_{k=0}^{\infty} \frac{(\mp s)^k}{(k-p)k!} \frac{d^k F(t(s), h(s))}{ds^k}. \quad (4)$$

It follows that there exists a unique pair of numbers $p^\pm(\mathcal{C})$ such that $A^\pm(\mathcal{C})=0$ for $p < p^\pm(\mathcal{C})$ and $A^\pm(\mathcal{C})=\infty$ for $p > p^\pm(\mathcal{C})$. The dependence on the particular curve has been indicated explicitly. This suggests the definition

$$p^\pm(\mathcal{C}) = \sup\{p \in \mathbb{R} | p \geq 1, A^\pm(\mathcal{C}) < \infty\} \quad (5)$$

for the *generalized order* of the phase transition along the curve \mathcal{C} . The difference $\Delta A = |A^+ - A^-|$ might be used to characterize the *strength* of the transition. For the Ising transition along the temperature direction one has $p^+ = p^- = 2 - \alpha$, where α is the specific-heat exponent. Similarly the order along the critical isotherm ($t=0$) is found to be $1 + 1/\delta$ by exchanging the role of t and h . Depending on the nature of the critical point the order may or may not be path dependent.

The considerations above show that analytic continuation of Ehrenfest's thermodynamic theory leads directly to power laws with nonclassical exponents. In addition, thermodynamic multiscaling arises because both amplitudes and exponents are in general path dependent. Equation (4) shows that the singular part of the free energy has the form

$$F(t(s), h(s)) = Y(\mathcal{C}) |s|^{p(\mathcal{C})}, \quad (6)$$

where Y and p are functions of the parameters defining the curve \mathcal{C} . Equation (6) represents a very general form of power-law scaling and includes general forms of multiscaling.

The last statement is readily verified by expanding the curve \mathcal{C} around $s=0$. Consistency with Eq. (6) suggests to require that \mathcal{C} can be expanded into a Frobenius series having the general form $|t(s)| = t_0 |s|^{\lambda_t} + \mathcal{O}(|s|^{\lambda_t+1})$ and $|h(s)| = h_0 |s|^{\lambda_h} + \mathcal{O}(|s|^{\lambda_h+1})$. Obviously, in the limit $s \rightarrow 0$ only the first term depending on the parameters t_0 , h_0 , λ_t , and λ_h remains important. To obtain the behavior of F in the t direction near the critical point one solves $|t(s)|$ and inserts the result into (6) and into $|h(s)|$. This gives $|h| = a |t|^\Delta$ near the critical point with $a = h_0 t_0^{-\Delta}$ and $\Delta = \lambda_h / \lambda_t$. Thus the free energy obeys the generalized scaling law

$$F(t, h) = Z(a, \Delta) |t|^{q(a, \Delta)}, \quad (7)$$

where $q(x, y)$ and $Z(x, y)$ are arbitrary functions. From Eq. (7) arise two natural forms of multiscaling if either one of the parameters a or Δ is expressed as a function of t and h . In the first case the scaling variable is $a = |h|/|t|^\Delta$; in the second case $\Delta = \log(|h|/a) / \log|t|$. While the first scaling variable has been used traditionally the second has only recently appeared in the multiscaling context.

Equations (6) and (7) are the main results of this pa-

per. Equation (6) shows that in a rather general way the concept of scaling is already implicitly present in traditional thermodynamics. Correspondingly Eq. (7) sheds new light onto the questions of scaling and universality from a thermodynamic point of view. While standard scaling is recovered as the special case in which q is constant multiscaling with nonuniversal critical exponents can occur in the general case. Moreover, according to (7) the characterization of universality classes for critical behavior may in general require two scaling functions rather than one scaling function combined with two characteristic critical exponents. Incidentally, the occurrence of two characteristic functions is also predicted from an *exact* renormalization-group treatment [17] for the free-energy density suggesting that traditional universality is only an approximate concept.

The critical exponents from the theory of critical phenomena can be related to the values of the function $q(x, \Delta)$ for $x \rightarrow 0^\pm$. One finds

$$\begin{aligned} 2 - \alpha &= \lim_{x \rightarrow 0^+} q(x, \Delta), & 2 - \alpha' &= \lim_{x \rightarrow 0^-} q(x, \Delta), \\ \beta &= \lim_{x \rightarrow 0^-} q(x, \Delta) - \Delta, & 1 + \frac{1}{\delta} &= \frac{1}{\Delta} \lim_{x \rightarrow \infty} q(x, \Delta), \\ \gamma &= 2\Delta - \lim_{x \rightarrow 0^+} q(x, \Delta), & \gamma' &= 2\Delta - \lim_{x \rightarrow 0^-} q(x, \Delta). \end{aligned}$$

If the function $q(x, \Delta)$ is discontinuous at $x=0$ then, e.g., $\alpha \neq \alpha'$ becomes possible. The standard scaling relations are recovered when q is constant.

Summarizing, this paper has shown that continuous phase transitions can be usefully classified according to their generalized orders. The classification implies new forms of thermodynamic multiscaling. The fact that multiscaling appears naturally in the general context of thermodynamics suggests that it may be a much more widespread phenomenon than currently apparent.

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