Lattice Quantum Hadrodynamics

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Quantum corrections to the mean-field equation of state for nuclear matter are estimated in a lattice simulation of quantum hadrodynamics. In contrast with the standard coordinate space methods used in lattice QCD, the calculations are carried out here in momentum space and on nonhypercubic (irregular) lattices. The quantum corrections to the known mean-field equation of state were found to be considerable.

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Wilson's fundamental paper [1] and the pioneering work done by Creutz and others in the late 1970s [2–4] paved the way to a numerical treatment of quantum chromodynamics on finite space-time lattices. In the following decade, the field of lattice QCD experienced a big boom [5]. Similar techniques were then applied to other quantum field theories, such as QED [6] or scalar ϕ^4 [7]; the present paper reports on the first application of lattice Monte Carlo methods to the theory of quantum hadrodynamics (QHD) [8,9].

Quantum hadrodynamics is a renormalizable, relativistic quantum field theory, designed for the description of the hadronic phase of nuclear matter. During the last decade, it has become one of the most widely used nuclear models, and was successfully applied to many problems in nuclear and astrophysics [9]. However, most of the work done in the framework of this theory so far was based on the mean-field approach (cf. [9]), and a major motivation for the present investigation was to find out "how good" the mean-field approximation actually is. More precisely, we wish to estimate quantum corrections to the mean-field equation of state for nuclear matter.

The straightforward transcription of quantum hadrodynamics on a lattice, using the standard finite-difference expressions for the kinetic terms, Wilson's method [10] to avoid fermion doubling, and the prescription given in [11] for the introduction of a chemical potential, led to intolerably high discretization errors already for the decoupled theory of a free Fermi gas [12]. A relative error of 20%–30% on a 20^4 hypercubic lattice would mean a deviation of several 100 MeV in the energy per nucleon [13], which was unacceptable. (A discretization error of the same order of magnitude was found in a similar investigation concerned with lattice QCD [14].)

The numerical error for the decoupled theory could be reduced by 1 order of magnitude after passing to a momentum-space formulation [12], and at the same time we had automatically solved the fermion doubling problem [10,15]. The usual coordinate approximation

$$\partial_\mu \psi(x) \rightarrow \frac{1}{2\Delta x}(\psi_{x+\Delta x \hat{e}_\mu} - \psi_{x-\Delta x \hat{e}_\mu})$$

is equivalent to

$$-ip_\mu \tilde{\psi}(p) \rightarrow -i\frac{\sin(\Delta x\, p_\mu)}{\Delta x}\tilde{\psi}(p)$$

in momentum space. Fermion doubling arises from the zeros of the sine at $p_\mu = \pm \pi/\Delta x$. In a momentum-space lattice formulation, p_μ arises instead of the sine term, vanishing only at the origin as it should be. Moreover, the dominating part of the discretization error, originating from the difference between $\sin(\Delta x\, p_\mu)/\Delta x$ and p_μ, disappears.

The Euclidean action for quantum hadrodynamics in momentum representation reads

$$S^E = \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\bar{\psi}(p)[(-i\gamma_a p_a + M)(2\pi)^4\delta^4(p-p') + ig_v\gamma_a V_a(p-p') - g_s\phi(p-p')]\psi(p')$$
$$+\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\phi(-p)[p_a p_a + m_s^2]\phi(p) + \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}V_a(-p)[(p_\beta p_\beta + m_v^2)\delta_{aa'} - p_a p_{a'}]V_{a'}(p)\,, \quad (1)$$

where ψ, ϕ, and V_a are the Fourier transforms of the nucleon, the scalar boson, and the vector boson field, respectively. Note that the reality of $\phi(x)$ and $V_a(x)$ implies that $\phi(-p) = \bar{\phi}(p)$ and $V_a(-p) = \bar{V}_a(p)$. (Complex conjugation is denoted by $z \rightarrow \bar{z}$.)

Passing to a discrete notation, we obtain (using the summation convention of Einstein)

$$S^E_{\text{lattice}} = \bar{\psi}_p (M_\psi)_{pp'}\psi_{p'} + \tfrac{1}{2}\phi_{-p}(M_\phi)_{pp'}\phi_{p'} + \tfrac{1}{2}V^a_{-p}(M_V)^{aa'}_{pp'}V^{a'}_{p'} \quad (2)$$

with the matrices

$$(M_\psi)_{pp'}=\left[(-i\gamma_a p_a+M)(2\pi)^4\frac{\delta_{pp'}}{(\Delta p)^4}+ig_v\gamma^a V^a_{p-p'}-g_s\phi_{p-p'}\right]\frac{(\Delta p)^4}{(2\pi)^4}\frac{(\Delta p')^4}{(2\pi)^4}, \tag{3}$$

$$(M_\phi)_{pp'}=[p_a p_a+m_s^2]\delta_{pp'}\frac{(\Delta p)^4}{(2\pi)^4}, \tag{4}$$

$$(M_V)^{aa'}_{pp'}=[(p_\beta p_\beta+m_v^2)\delta^{aa'}-p^a p^{a'}]\delta_{pp'}\frac{(\Delta p)^4}{(2\pi)^4}. \tag{5}$$

The free action of a scalar boson ϕ of mass m_s and a vector boson V_a of mass m_v, as well as the Yukawa couplings to a fermion field ψ with mass M, are easily recognized. The γ_a are Euclidean Dirac matrices [15] and $(\Delta p)^4$ denotes the elementary lattice volume element. Dirac and isospin indices are suppressed. A chemical potential can be included in the "naive" way [11] by adding $-\mu\gamma_4\delta_{pp'}(\Delta p)^4/(2\pi)^4$ to the fermion matrix M_ψ. Euclidean metric is understood throughout, $p_a p_a=p_1^2+\cdots+p_4^2$.

The obvious disadvantage of the momentum-space formulation is the nonlocal interaction term: The original (space-time) expression

$$\int d^4x\,\bar\psi(x)\phi(x)\psi(x) \tag{6}$$

has turned into

$$\int\int\frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\bar\psi(p)\phi(p-p')\psi(p'), \tag{7}$$

and similarly for the vector boson. On the lattice, this leads to a fully occupied fermion matrix M_ψ. Its inverse and determinant have to be calculated iteratively during the Monte Carlo process, and this will be the most time-consuming step in the numerical procedure.

Besides the advantages already mentioned, the momentum-space formulation also allows for an easy reduction to mean-field theory, by just suppressing all nonzero Fourier components of the meson fields. Then the fermion matrix is block diagonal (diagonal in p) and a mean-field lattice calculation with only five nonzero meson field components ($\phi_{(p=0)}$ and $V^a_{(p=0)}$) is easy to do. In this way, the analytically known mean-field results for quantum hadrodynamics could be reproduced to within $\sim 1\%$ in a cold start, self-consistent Monte Carlo calculation.

The significance of this mean-field lattice calculation is twofold. First, the same computer code is used as for the full calculation, except that the majority of meson field components is set to zero. This provides a convenient check of the whole numerical procedure. Second, we will define the quantum corrections to the mean-field equation of state to be the difference between the *lattice results* with the nonzero Fourier components of meson fields switched on and off, respectively. In this way, one can hope to reduce that part of the systematic error which is due to the lattice geometry.

Two more points deserve a comment before we come to a discussion of the numerical results.

(1) An irregular (nonhypercubic) lattice geometry turned out to be favorable. In particular, we chose a "spherically symmetric" distribution of the spatial momenta inside and outside the Fermi sphere, and used a cubic distance law in the timelike direction. It follows from (7) that the meson fields must be known for all differences $p-p'$, where p and p' are any two points of the baryon lattice. Hence, an irregular baryon lattice implies a meson lattice of the size (No. of baryonic lattice points)$^2/|\mathcal{G}|$, where $\mathcal{G}$ is the symmetry group of the baryon lattice [16].

(2) In order to avoid ultraviolet divergencies, the negative-energy baryon states are eliminated. This can be achieved in a lattice formulation by subtracting from the effective action

$$S_{\rm eff}=S_{\rm free}[\phi,V_a]-{\rm Tr}\ln M_\psi^{(\mu)}[\phi,V_a] \tag{8}$$

as well as from the baryonic observables

$${\rm Tr}\{\mathcal{O}(M_\psi^{(\mu)}[\phi,V_a])^{-1}\}, \tag{9}$$

the respective expressions at vanishing chemical potential $\mu=0$.

This method of regularization is applied to both the mean-field and the full calculation. Thus, our lattice results will reflect many-body effects of an interacting Fermi gas of nucleons to all orders, take into account the correct relativistic kinematics, but will neglect the influence of the negative-energy baryon spectrum.

Starting from a 576-point, irregular baryon lattice with cutoff $1.5k_F$, we obtained a meson lattice with ~ 83000 nodes and cutoff $3k_F$, on which a Monte Carlo simulation of the scalar and the vector field was carried out. We used the standard Metropolis algorithm [17,18] with a particular adaption to the momentum-space representation which is described below.

The average Monte Carlo hit amplitude was adjusted *locally*, i.e., separately for each lattice point in such a way that an average acceptance rate of 0.5 was obtained.

To determine the change in the effective action (8) that would result from

$$\varphi_j\to\varphi_j+\Delta\varphi_j \quad \text{for } \varphi_j=\phi_q, V^a_q \tag{10}$$

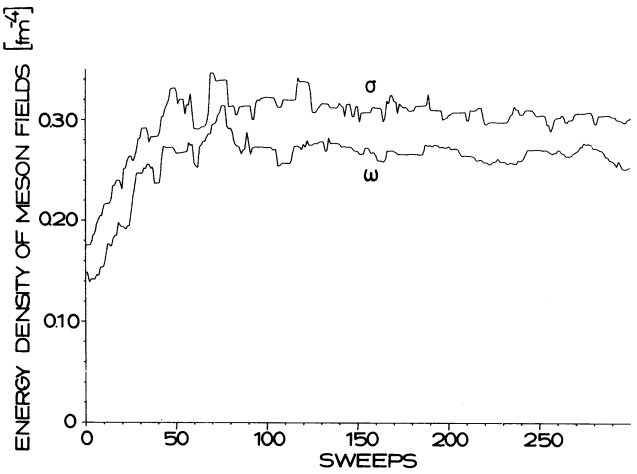

FIG. 1. Mesonic contributions to the energy density during a 300 sweep production run. (The contribution from the sigma meson is plotted with the opposite sign.)

the partial derivatives

$$\frac{\partial}{\partial\varphi_j}S_{\text{eff}} \tag{11}$$

were calculated for *all* field variables before a sweep, and multiplied by $\Delta\varphi_j$ at the time of the sweep. For the free, quadratic meson action, the evaluation of (11) is trivial, and for the fermion determinant we used the identity

$$\frac{\partial}{\partial\varphi_j}\operatorname{Tr}\ln M_\psi[\varphi]=\operatorname{Tr}\left[\frac{\partial M_\psi}{\partial\varphi_j}M_\psi^{-1}[\varphi]\right]. \tag{12}$$

The exact inverse M_ψ^{-1} was calculated before each sweep, and the trace in (12) is easy to evaluate since $\partial M_\psi/\partial\varphi_j$ is a sparse matrix. With a single update of M_ψ^{-1} before the sweep, the Monte Carlo process is obviously more reliable for the field variables which are treated first than for those treated last. In order to compensate this effect, the sequence of the φ_j during a sweep was randomized. The quality of this approximation has been tested by comparing to runs with more frequent updates of the inverse, and it was found to be reasonably good.

The lattice action, as well as other observables (baryon energy and density, meson energies), became stable after $\sim$100 warm-up sweeps (cf. Fig. 1), the runs were then continued another 200 sweeps for the measurements. Since the sampled observables after the initial warm-up became quite stable, this relatively small number of measurements was enough to obtain a sufficiently small statistical error.

All calculations were carried out in the partially quenched approximation, neglecting the imaginary part of the effective action in the Metropolis algorithm. Attempts to include it by grouping it with the observables, i.e., sampling

$$\mathcal{O}[\varphi]e^{i\,\mathrm{Im}S_{\text{eff}}[\varphi]} \tag{13}$$

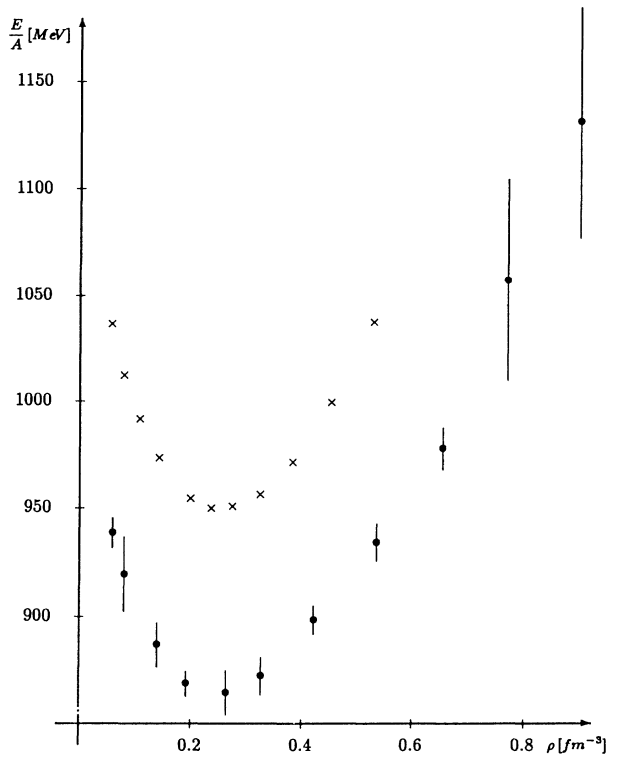

FIG. 2. Quantum corrections for the binding energy per nucleon. Lattice simulation in momentum space. The crosses represent the mean-field results, the errors are statistical only.

instead of $\mathcal{O}[\varphi]$ alone, led to similar results as in the partially quenched case up to a reasonably high number of sweeps. Eventually, however, increasing oscillations originating from numerical instabilities prevented us from obtaining the desired statistics.

The observables baryon density ρ_B, baryon energy density ϵ_B, sigma energy density ϵ_σ, and omega energy density ϵ_ω were sampled in 11 lattice runs with 300 sweeps (100 warm-ups) each, and the energy per nucleon was calculated as

$$E/A=(\epsilon_B+\epsilon_\sigma+\epsilon_\omega)/\rho_B\,. \tag{14}$$

The result is shown as a function of ρ_B in Fig. 2.

The first important observation is that the fully interacting system still saturates. Comparing with the mean-field calculation we observe that the quantum fluctuations lead to an additional saturation energy of about 85 MeV, while the saturation density remains about the same. Note that we take the *lattice* mean-field calculation as a reference, because in this way we hope to exclude at least part of the systematic error, caused by the lattice geometry. The fact that the inclusion of quantum fluctuations leads to additional binding is what one expects from the variational principle. The amount of the decrease certainly represents a lower limit, but how much of it was lost due to restrictions of the lattice size is one of

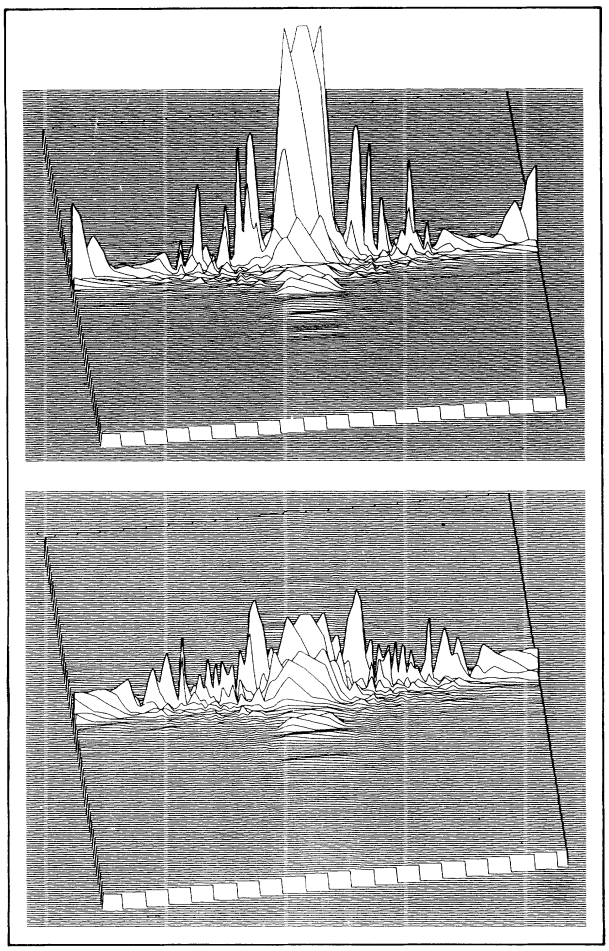

FIG. 3. The interacting vector meson field (first and second components) after 250 sweeps. The elevated rectangle shows the extension of the lattice. The spatial momentum is on the horizontal axis, the timelike component in the depth. The cutoffs are 828.8 MeV in the spacelike and 28.6 GeV in the timelike direction.

the most important questions to be investigated in the future. Looking at the meson fields during run time, however, gives a clear indication that the largest part of the quantum effects was included: Snapshots of the field configurations have been taken for all meson fields at various stages of the Monte Carlo run and examples are given in Fig. 3. Similar pictures were obtained for all field components and at all times. Obviously, some field strength is cut off in the spacelike direction, but the dominating part is clearly included.

Taking into account the "natural" cutoff of the nuclear interaction at ~ 1 GeV due to the finite nucleon size (which is not correctly included in quantum hadrodynamics) a considerable extension of the momentum cutoff seems to be unphysical. To verify this point, calculations on larger lattices including nucleon form factors are planned.

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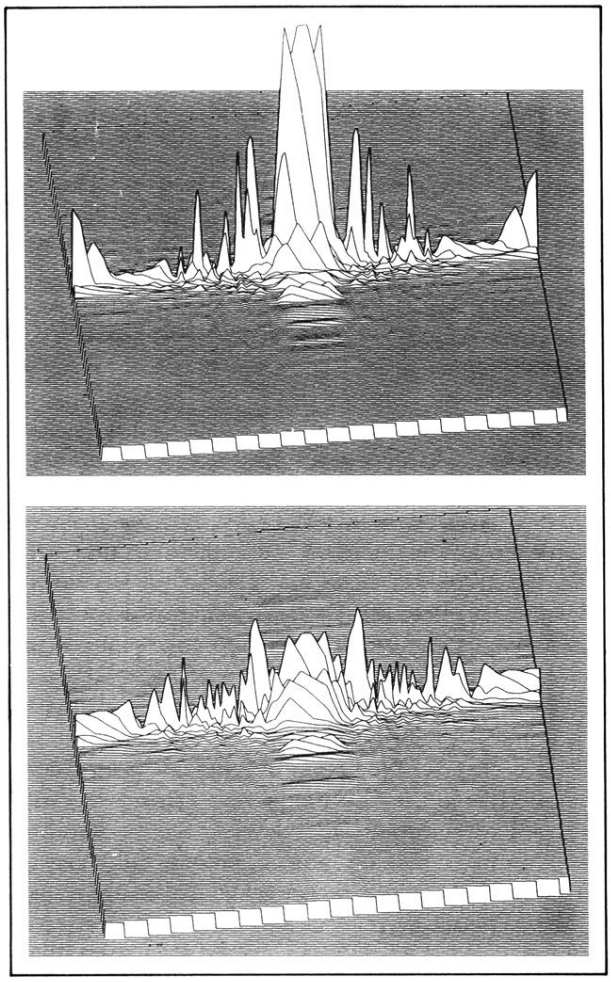

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