Evidence for a Yang-Mills Fractal

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(Received 5 June 1991; revised manuscript received 23 September 1991)

A pure (2+1)-dimensional classical Yang-Mills system is started with a sine-wave nonuniformity in space. One of the potentials soon develops, as a function of position, into an approximate fractal with fractal dimension 1.40.

PACS numbers: 11.15.Kc, 05.45.+b, 11.10.Lm

Chaotic behavior in time was first described in 1981, by Matinyan, Savvidi, and Ter-Arutyunyan-Savvidi [1] in the case of Yang-Mills systems whose spatial behavior was kept very simple. In a recent numerical study [2], the present author had envisaged a spatial counterpart of that behavior (perhaps a fractal) in the case of time-independent Yang-Mills potentials in 2+1 dimensions, with a point source at the origin. No fractals were found under those conditions. However, in the absence of external sources, an approximate fractal behavior can emerge after some time if the system is allowed to evolve from a spatially nonuniform but smooth beginning.

In order to demonstrate this we consider, in 2+1 dimensions, the SU(2) potentials

$$A^{\mu}=\tfrac{1}{2}\sigma_a A_a^{\mu}\quad(\mu=0,1,2;\,a=1,2,3)\,, \tag{1}$$

where the A_a^{μ} are real functions of space-time and the σ_a are the Pauli matrices. We solve the Yang-Mills equations

$$D_{\mu}F^{\mu\nu}=0\,, \tag{2}$$

where

$$F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}-ie[A^{\mu},A^{\nu}]\,, \tag{3}$$

$$D_{\mu}=\partial_{\mu}-ie[A_{\mu},\ ]\,, \tag{4}$$

with a coupling constant e.

Denoting the three dimensions by t,x,y, we choose the starting nonuniformity to be in the x direction, so that the initial conditions are still uniform in y. The y uniformity will persist, and thus we only need to solve differential equations in 1+1 dimensions. At the same time, the near triviality of a pure (1+1)-dimensional Yang-Mills system is being avoided.

Based on the temporal gauge $A^0=0$, with $A^1=A_x$, $A^2=A_y$, we have the fields

$$E_x=-\partial_0 A_x\,,\quad E_y=-\partial_0 A_y\,, \tag{5}$$

$$B=\partial_x A_y+ie[A_x,A_y]\,, \tag{6}$$

obeying

$$\partial_x E_x+ie[A_x,E_x]+ie[A_y,E_y]=0\,, \tag{7}$$

$$\partial_t E_x-ie[A_y,B]=0\,, \tag{8}$$

$$\partial_t E_y+\partial_x B+ie[A_x,B]=0\,. \tag{9}$$

In this gauge there remain six potential functions to determine. But there exists a reduced solution, involving only three functions: We base the rest of this paper on the self-consistent choice

$$A_x=\tfrac{1}{2}\sigma_1 v\,, \tag{10}$$

$$A_y=\tfrac{1}{2}\sigma_2 w+\tfrac{1}{2}\sigma_3 u\,, \tag{11}$$

u,v,w being real functions of t and x. The Yang-Mills equations reduce to

$$(\partial_t^2-\partial_x^2)u+e(2v\,\partial_x w+w\,\partial_x v)+e^2v^2u=0\,, \tag{12}$$

$$(\partial_t^2-\partial_x^2)w-e(2v\,\partial_x u+u\,\partial_x v)+e^2v^2w=0\,, \tag{13}$$

$$\partial_t^2 v+e(u\,\partial_x w-w\,\partial_x u)+e^2(u^2+w^2)v=0\,, \tag{14}$$

$$\partial_x\partial_t v+e(u\,\partial_t w-w\,\partial_t u)=0\,. \tag{15}$$

Equation (15) is Gauss's law, which can be viewed as a constraint on v. This equation serves as a valuable consistency and convergence check throughout the numerical calculations.

The result of interest to us here is elicited by initial conditions of the form

$$v=w=0,\quad u=a\,, \tag{16}$$

$$\partial_t v=abe\cos x,\quad \partial_t w=b\sin x,\quad \partial_t u=0 \tag{17}$$

(a,b constant); they are constructed so as to respect Eq. (15). The numerical results shown below are for a large coupling, $e=64$. We take $a=b=1$. The large values of e and of $\partial_t v$ are designed to encourage nonlinear behavior.

Figure 1 shows $w(x)$, selected for its fractal appearance. In the figure, time equals 0.294. (This is as far as computing time has taken us.) From its single-mode beginning, w has now acquired the wave-number spectrum shown in Fig. 2.

The fractal dimension of Fig. 1 was evaluated by the box-counting method, starting with a box that just encompasses the curve itself. The resulting log-log plot is shown in Fig. 3; the first six points of the plot yield a slope equal to 1.40.

The time dependence of u,v,w (not shown) was calculated at representative values of x ($x=25°,45°,65°$) and turns out to be quite smooth between $t=0$ and $t=0.294$.

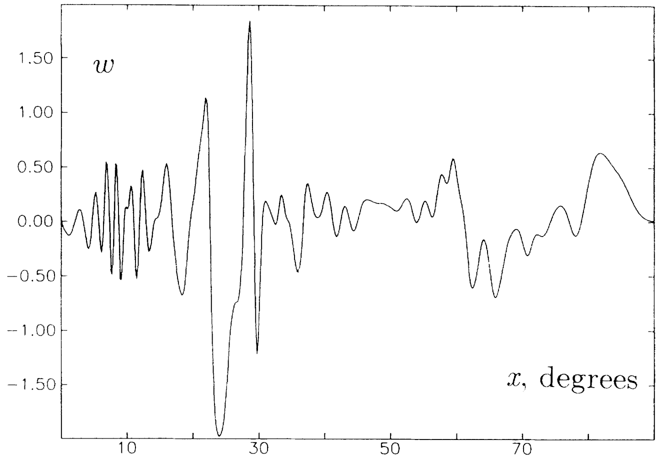

FIG. 1. A Yang-Mills potential $w(x)$, whose initial conditions are given by (16) and (17), is shown here for time $t=0.294$ (x is given in degrees). The zero at 90° is presumably not exact.

To conclude, in the sense that a fractal dimension can be calculated, a pure classical Yang-Mills system can develop an approximately fractal behavior from smooth beginnings. That result raises several questions, some of which should become answerable through substantially expanded computing time. In particular, there are the following.

(1) Will the approach to fractals turn out to be the rule in wide classes of nonlinear field theories?

(2) How does the fractal dimension, here 1.40, depend on the parameters of the model, and does it reach a limit in the course of time?

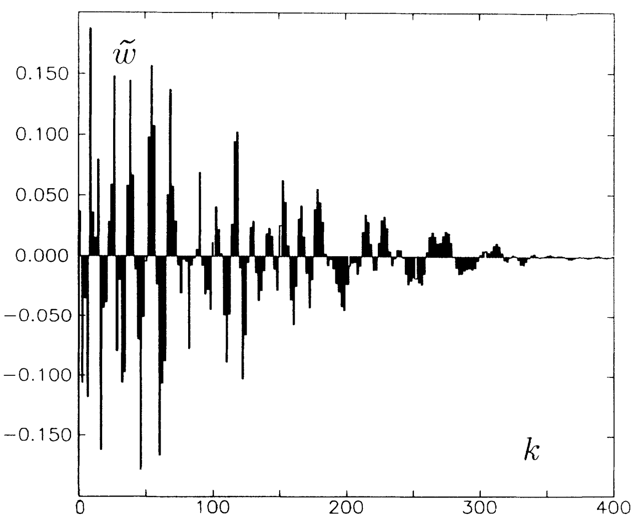

FIG. 2. The spectrum $\tilde{w}(k)$ at $t=0.294$. Its relation to Fig. 1 is $w(x)=\sum_{k=\mathrm{odd}}\tilde{w}(k)\sin kx$. The calculational cutoff is $k<600$.

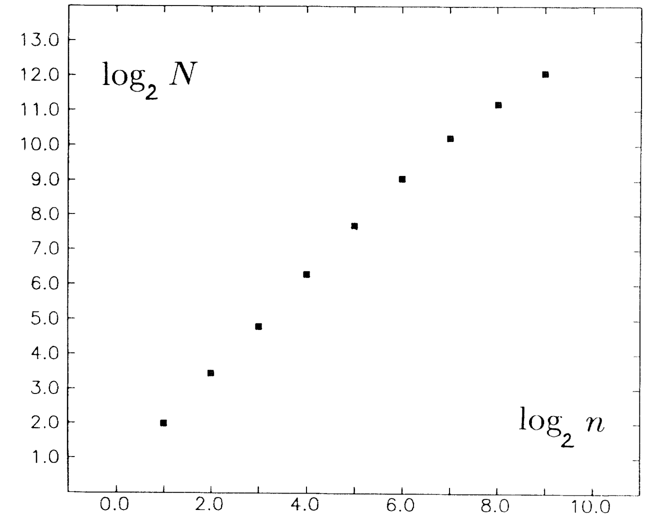

FIG. 3. Plot of $\log_2 N/\log_2 n$, when the curve of Fig. 1 is covered with an $n\times n$ grid of rectangles, and N is the number of rectangles containing a point of the curve. The time is $t=0.294$. The "knee in the curve" near $\log_2 n=6$ is a transition to unit slope, as the grid begins to detect a smooth graph.

(3) Is this discussion relevant to cosmological models? If, at any time during its evolution, the Universe was dominated by a nonlinear classical field, a mechanism for fractal structure should be envisaged. Such a structure has been proposed several times on observational and theoretical grounds [3,4]. Here we have been concerned with an approximate fractal that exists in a space of which one (ordinary) dimension is the value, w, of a potential. For cosmology we need a fractal that exists in ordinary three-space. How the first kind can induce the second kind is discussed by Mandelbrot [3] in the context of sections through fractals.

Some remarks on reliability and calculational limits.—The six featured log-log points are unexpectedly well aligned for this type of calculation. To enable a comparison we include Table I, which displays the log-log data for the somewhat earlier time, $t=0.274$, where such a de-

TABLE I. Data for the log-log plot at two times.

	$\log_2 N$	
$\log_2 n$	$t=0.274$	$t=0.294$
1	2.00	2.00
2	3.69	3.45
3	4.99	4.80
4	6.35	6.29
5	7.82	7.70
6	9.12	9.06
7	10.28	10.22
8	11.27	11.19
9	12.16	12.08

gree of alignment has not yet been reached, and where the sixth point is not yet firm.

The accuracy of the result is gauged by the extent to which Gauss's law, Eq. (15), is numerically satisfied. At $t=0.294$, Gauss's law begins to deteriorate when examined on a fine scale, namely, beyond the first 120 nonzero Fourier modes. This resolution must be compared with the detail needed to evaluate the sixth point in Fig. 3, namely, $2^6=64$ samplings along the x axis in Fig. 1, yielding a safe margin.

What is the outlook on further calculations? A hypothetical linear extension to a seventh point would require 128 samplings, at the edge of present reliability; sharply increasing the computation time would overcome that limitation. (The present results have used over two hours of CPU time on the Pittsburgh Supercomputer's Cray YMP.)

I have benefited from several informative discussions with Professor Harvey Kaplan. Thanks are due to the Pittsburgh Supercomputing Center for computer support. I am indebted to Dr. Wayne Fordyce and Jack Zhu for help in improving the program's efficiency, as well as adapting it to the Cray YMP machine.

[1] For a review, see S. G. Matinyan, Fiz. Elem. Chastits At. Yadra **16**, 522 (1985) [Sov. J. Part. Nucl. **16**, 226 (1985)]; some additional articles are listed in Ref. [2].

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[4] E. Vishniac, K. Olive, and D. Seckel, Nucl. Phys. **B289**, 717 (1987); P. H. Coleman, in *Fractals' Physical Origin and Properties*, edited by L. Pietronero (Plenum, New York, 1989).