Canonical Formulation of the Self-Dual Yang-Mills System: Algebras and Hierarchies

Ling-Lie Chau[(1),(2)] and Itaru Yamanaka[(1)]
[(1)]*Department of Physics, University of California at Davis, Davis, California 95616*
[(2)]*Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
(Received 10 April 1991)

We construct a canonical formulation of the self-dual Yang-Mills system formulated in the gauge-invariant group-valued J fields and derive their Hamiltonian and the quadratic algebras of the fundamental Dirac brackets. We also show that the quadratic algebras satisfy Jacobi identities and their structure matrices satisfy modified Yang-Baxter equations. From these quadratic algebras, we construct Kac-Moody-like and Virasoro-like algebras. We also discuss their related symmetries, involutive conserved quantities, and hierarchies of nonlinear and linear equations.

PACS numbers: 11.15.−q

In recent years, classical self-dual Yang-Mills (SDYM) equations have emerged as a beautiful and powerful mathematical system, possessing beautiful solutions of instantons [1] and monopoles [2,3] and serving as an essential tool for analyzing exotic four-manifolds [4]. The SDYM system also possesses many traits of integrable systems such as linear systems [5], infinite nonlocal conservation laws [6], and Bäcklund transformations. Further, many two-dimensional (2D) systems are shown to be reductions from the four-dimensional (4D) SDYM equations: sine-Gordon, Korteweg-de Vries, nonlinear Schrödinger equation, Liouville equation [7], chiral models [8], and chiral model with arbitrary Wess-Zumino term [9]. When properly formulated, the 4D full supersymmetric $n \geq 3$ Yang-Mills equations [10] and the full supersymmetric $n \geq 5$ conformal supergravity [11] all resemble SDYM equations. Thus, the SDYM system is becoming a crossway between 2D and 4D systems from the integrable-system point of view. All the studies so far are mainly classical. Though the 4D theories possess many traits of the integrable system of 2D theories, they certainly will not and should not turn out to be integrable in the same way as the 2D theories. However, these beautiful characteristics of integrable systems should be made best use of in a quantum field theoretical way. Perhaps in the process we will find a nonperturbative approach to the 4D quantum field theories. Here we shall present some useful results in this direction.

In studying the SDYM system, we exploit the fact that in the J formulation the SDYM equations and action resemble those of the Wess-Zumino-Novikov-Witten (WZNW) model. Following the procedure developed in Ref. [12] for the WZNW model, we construct a canonical formulation for the SDYM J fields [13]. We derive explicitly the Hamiltonian and the fundamental Dirac brackets for J fields that form quadratic algebras. We also show that the quadratic algebras satisfy Jacobi identities and their structure matrices, which contain dynamical variables, satisfy modified Yang-Baxter equations. Following naturally from the quadratic algebras, we construct Kac-Moody-like algebras, Virasoro-like algebras, and their corresponding hierarchies of nonlinear equations and linear systems.

Several important features emerge from this 4D theory: The Hamiltonian is an interactive one; in contrast, the Hamiltonian of the 2D WZNW model is free. The J field can no longer be factorized into the so-called "left"-chiral and "right"-chiral fields as in the WZNW model. There are four distinct sets of algebras as compared to two in the WZNW model. Among the algebras, quadratic algebras are the most fundamental and will serve as a basis for the future development of quantum field theory [14,15]. (Our formulation here does not deal with the globally nontrivial sectors of the SDYM theory.)

Self-dual Yang-Mills equations in the J fields [3,16,17].—We consider the SDYM equations $F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ ($\mu,\nu,\rho,\sigma = 1,2,3,4$), where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ in a four-dimensional space with a diag$(+,-,+,-)$ metric. Using the light-cone coordinates, $y \equiv x^1 + x^2$, $\bar{y} \equiv x^1 - x^2$, $z \equiv x^3 - x^4$, and $\bar{z} \equiv x^3 + x^4$, the SDYM equations become $F_{yz} = 0$, $F_{\bar{y}\bar{z}} = 0$, and $F_{y\bar{y}} + F_{z\bar{z}} = 0$. From the first and second, curvatureless equations, the gauge potentials have the following representation: $A_y = D^{-1}\partial_y D$, $A_z = D^{-1}\partial_z D$, $A_{\bar{y}} = \bar{D}^{-1}\partial_{\bar{y}}\bar{D}$, and $A_{\bar{z}} = \bar{D}^{-1}\partial_{\bar{z}}\bar{D}$. For general $n \times n$ matrices D, the gauge group is GL(N). Without loss of generality, we can choose $A_{\bar{y}}$ and $A_{\bar{z}}$ to be zero, which can be achieved by the transformations $A_M \rightarrow B_M = \bar{D}A_M\bar{D}^{-1} + \bar{D}\partial_M\bar{D}^{-1}$, $M \equiv y,z,\bar{y},\bar{z}$. The nonzero components of the gauge fields are $A_y \rightarrow B_y = J^{-1}\partial_y J$, $A_z \rightarrow B_z = J^{-1}\partial_z J$, where $J \equiv D\bar{D}^{-1}$ and is gauge invariant. From the third self-dual equation we obtain the equations of motion for the fields $J(y,z,\bar{y},\bar{z})$, $\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) = 0$, which we call the "right" SDYM J field equations.

There is another equivalent set of equations. By choosing $A_y, A_z \rightarrow 0$ and $A_{\bar{y}} \rightarrow \bar{B}_{\bar{y}} = J\partial_{\bar{y}}J^{-1}$, $A_{\bar{z}} \rightarrow \bar{B}_{\bar{z}} = J\partial_{\bar{z}}J^{-1}$, the equations of motion are then replaced by what we call the "left" equations, $\partial_y(J\partial_{\bar{y}}J^{-1}) + \partial_z(J\partial_{\bar{z}}J^{-1}) = 0$, which are just the right J field equations with the changes $J \leftrightarrow J^{-1}$ and $y,z \leftrightarrow \bar{y},\bar{z}$, respectively.

Symmetries of SDYM J field equations [3,16,17].

—The SDYM equations for the J fields are invariant under the transformations $J\to\bar V(\bar y,\bar z)JV^{-1}(y,z)$, where $V(y,z)$ and $\bar V(\bar y,\bar z)$ are arbitrary matrix functions. The symmetry generated by $V(y,z)$ can be understood as residual gauge transformations for B_M, and that by $\bar V(\bar y,\bar z)$ as the gauge transformations for $\bar B_M$. These symmetries are analogous to those in the WZNW model, $g\to\bar V(\bar y)gV(y)$.

A canonical formation for the SDYM J fields.— Following these hints of the analogy with the WZNW model, an action [13] that gives the SDYM J field equations can be constructed:

$$A=\alpha\Big[\int dy\,dz\,d\bar z\int_{-L/2}^{L/2}d\bar y\,\mathrm{tr}(\partial_y J\,\partial_{\bar y}J^{-1})-\int dy\,dz\,d\bar z\int_{-L/2}^{L/2}d\bar y\int_0^1 d\rho\,\mathrm{tr}(J^{-1}\partial_\rho J[J^{-1}\partial_{\bar y}J,J^{-1}\partial_y J])$$
$$+\int dy\,dz\,d\bar z\int_{-L/2}^{L/2}d\bar y\,\mathrm{tr}(\partial_z J\,\partial_{\bar z}J^{-1})-\int dy\,dz\,d\bar z\int_{-L/2}^{L/2}d\bar y\int_0^1 d\rho\,\mathrm{tr}(J^{-1}\partial_\rho J[J^{-1}\partial_{\bar z}J,J^{-1}\partial_z J])\Big]\,, \quad (1)$$

where α is a dimensional coupling constant.

Any one of the variables $y,z,\bar y,\bar z$ can be chosen as "time." We first pick y as time. We apply a periodic boundary condition in the "space" direction of $\bar y$, $J(y,\bar y+L/2,z,\bar z)=J(y,\bar y-L/2,z,\bar z)$, but we shall show that physical results are independent of L. When y is chosen as time, $\bar y$ (space) becomes special and the Poisson and Dirac brackets have a nonultralocal δ' term in $\bar y$, yet only an ultralocal δ term in the other two special coordinates $z,\bar z$. Here we follow the same procedure established in Ref. [12] and refer the reader to it for details.

The canonical momenta of the fields J is $\pi_{\alpha\beta}=\delta A/\delta(\partial_y J_{\alpha\beta})=\alpha(\partial_{\bar y}J^{-1}_{\beta\alpha}-e_{\beta\alpha})$, where $e_{\beta\alpha}$ are defined by

$$\Big[-\alpha\int dy\,dz\,d\bar z\int_{-L/2}^{L/2}d\bar y\,\mathrm{tr}(e\,\partial_y J)\Big]\equiv[\text{second term of Eq. (1)}]\,.$$

Equal-time (y) Poisson brackets are

$$\{J_{\alpha\beta}(y,\mathbf{x}),J_{\gamma\delta}(y,\mathbf{x}')\}_P=0,\quad \{\pi_{\alpha\beta}(y,\mathbf{x}),\pi_{\gamma\delta}(y,\mathbf{x}')\}_P=0\,,$$
$$\{\pi_{\alpha\beta}(y,\mathbf{x}),J_{\gamma\delta}(y,\mathbf{x}')\}_P=\delta_{\alpha\gamma}\delta_{\beta\delta}\sum_n\delta(\bar y-\bar y'+nL)\delta^2(Z-Z')\,, \quad (2)$$

where we used a space vector $\mathbf{x}\equiv(\bar y,z,\bar z)$ and $\delta^2(Z-Z')\equiv\delta(z-z')\delta(\bar z-\bar z')$. The constraints from the definition of canonical momenta are $C_{\alpha\beta}(\mathbf{x})=\pi_{\alpha\beta}-\alpha(\partial_{\bar y}J^{-1}_{\beta\alpha}-e_{\beta\alpha})$, which give an infinite number of constraints, one at each point of $\mathbf{x}$. We can rewrite the constraints in the following form:

$$E_{\rho\beta}\equiv J_{\alpha\rho}C_{\alpha\beta}=J_{\alpha\rho}\pi_{\alpha\beta}+\alpha[(J^{-1}\partial_{\bar y}J)_{\beta\rho}+(eJ)_{\beta\rho}]\,.$$

The Poisson brackets for $E_{\alpha\beta}$ are

$$\{E_{\alpha\epsilon}(y,\mathbf{x}),E_{\delta\gamma}(y,\mathbf{x}')\}_P=\Big[-2\alpha\delta_{\gamma\alpha}\delta_{\epsilon\delta}\sum_n\delta'(\bar y-\bar y'+nL)+(E^T_{\gamma\alpha}\delta_{\epsilon\delta}-\delta_{\gamma\alpha}E^T_{\epsilon\delta})\sum_n\delta(\bar y-\bar y'+nL)\Big]\delta^2(Z-Z')\,, \quad (3)$$

where the superscript T denotes the transpose. We can separate the constraints into the first-class ones and the second-class ones, $F^{\rho\sigma}=\int_{-L/2}^{L/2}d\bar y\,\lambda^{\rho\sigma}_{\alpha\gamma}(\mathbf{x})E_{\alpha\gamma}(\mathbf{x})\approx 0$, $S^{\rho\sigma}=\lambda^{\rho\sigma}_{\alpha\gamma}E_{\alpha\gamma}-L^{-1}F^{\rho\sigma}\approx 0$, respectively. The $\lambda^{\rho\sigma}_{\alpha\gamma}(\mathbf{x})$'s are unknown fields with the boundary condition $\lambda^{\rho\sigma}_{\alpha\gamma}(-L/2)=\lambda^{\rho\sigma}_{\alpha\gamma}(L/2)=\delta^\rho_\alpha\delta^\sigma_\gamma$ and determined solely by the conditions of first-class constraints, $\{F^{\rho\sigma},E_{\alpha\gamma}\}\approx 0$, which implies $\int_{-L/2}^{L/2}d\bar y\,\lambda^{\rho\sigma}_{\alpha\gamma}(\mathbf{x})\sum_{n=-\infty}^{\infty}\partial_{\bar y}\delta(\bar y-\bar y'+nL)=0$, with solution $\lambda^{\rho\sigma}_{\alpha\gamma}(\mathbf{x})=\lambda^{\rho\sigma}_{\alpha\gamma}(\bar y=0,\bar z,z)=\delta^\rho_\alpha\delta^\sigma_\gamma$, which gives $F^{\rho\sigma}(z,\bar z)=\int_{-L/2}^{L/2}d\bar y\,E^{\rho\sigma}(\mathbf{x})$.

The total Hamiltonian is then $H_T=-A_{\rm pl}+\int d^3x\times u_{\alpha\epsilon}E_{\alpha\epsilon}$, where $A_{\rm pl}$ is the integrand for the y integration of the third and fourth terms of the action, Eq. (1). The functions $u_{\alpha\epsilon}$ are solutions of the following consistency conditions: $\{H_T,E_{\delta\gamma}\}_P=0$, i.e., $\partial_{\bar y}u_{\alpha\epsilon}=-\partial_{\bar z}(J^{-1}\partial_z J)$, and $u_{\gamma\delta}=\chi_{\gamma\delta}+\Lambda_{\gamma\delta}$, where Λ is the homogeneous solution (thus independent of $\bar y$) and χ is the particular solution. Now the total Hamiltonian is

$$H_T=\Big[\int d^3x\,\chi_{\alpha\epsilon}E_{\alpha\epsilon}-A_{\rm pl}\Big]+\int d^3x\,\Lambda_{\gamma\delta}E_{\gamma\delta}\,. \quad (4)$$

The canonical equations of motion of this Hamiltonian can be shown to give Euler-Lagrange equations for the J fields, $\partial_y(J\partial_{\bar y}J^{-1})=\{H_T,J\partial_{\bar y}J^{-1}\}=-\partial_z(J\partial_{\bar z}J^{-1})$. Note that this Hamiltonian is interactive; in contrast, the Hamiltonian of the WZNW model is free, corresponding to having only the last term [12] in Eq. (4).

Similarly, we can choose $\bar y$ as time and y as space with periodic boundary condition and obtain Eqs. (2) and (4) with $J\leftrightarrow J^{-1}$ and $\bar y\leftrightarrow y$, $\bar z\leftrightarrow z$. We then can also choose z or $\bar z$ as times and obtain two other sets of quadratic algebras. [They are Eqs. (2) to Eq. (4) with the obvious corresponding changes.] So, in total, the 4D SDYM system is spanned by four different Hamiltonian quadratic algebras.

Quadratic algebras of Dirac brackets and Kac-Moody-like algebras.— From the canonical formulation given in the previous section, we can derive the following Dirac brackets which are quadratic algebras:

$$\{\tilde J_{\rm I}(\mathbf{x}),\tilde J_{\rm II}(\mathbf{x}')\}_D=\tilde J_{\rm I}(\mathbf{x})\tilde J_{\rm II}(\mathbf{x}')M_{\rm I,II}\,, \quad (5)$$

where

$$M_{\mathrm{I,II}} \equiv -(\mathbf{P}_{\mathrm{I,II}}/2\alpha)\left[\tfrac{1}{2}\sum_n \varepsilon(\bar{y}-\bar{y}'+nL)e^{nQ_{\mathrm{I,II}}}+(1/Q_{\mathrm{I,II}})\right]\delta^2(Z-Z'),$$

and $Q_{\mathrm{I,II}} \equiv (F_{\mathrm{I}}^T - F_{\mathrm{II}}^T)/2\alpha$; the superscript T denotes the transpose. The $\tilde{J}$ fields are related to the J fields by a zero-mode factor $\tilde{J}(\mathbf{x}) \equiv J(\mathbf{x})\exp(-\bar{y}F^T/2\alpha L)$. The bracket of any tensors $A_{\mathrm{I}} \equiv A\otimes I$ and $B_{\mathrm{II}} \equiv I\otimes B$ is defined as $\{A_{\mathrm{I}},B_{\mathrm{II}}\}_{\alpha\gamma,\beta\delta} \equiv \{A\otimes 1, 1\otimes B\}_{\alpha\gamma,\beta\delta} = \{A_{\alpha\beta},B_{\gamma\delta}\}$. The permutation matrix $\mathbf{P}_{\mathrm{I,II}}$ is defined through the expression $\mathbf{P}_{\mathrm{I,II}}(A_{\mathrm{I}}B_{\mathrm{II}})\mathbf{P}_{\mathrm{I,II}} = B_{\mathrm{I}}A_{\mathrm{II}}$, for any matrices A and B, and $\mathbf{P}_{\mathrm{I,II}}\mathbf{P}_{\mathrm{I,II}} = I$. The function $\varepsilon(x)$ stands for the sign of the argument x. We will see that the current formed from $\tilde{J}$, not J, satisfies the Kac-Moody-like algebra. Taking the limit $L\to\infty$, we obtain an L-independent result:

$$\lim_{L\to\infty} M_{\mathrm{I,II}} = -(\mathbf{P}_{\mathrm{I,II}}/2\alpha)[\tfrac{1}{2}\varepsilon(\bar{y}-\bar{y}') - \tfrac{1}{2}\coth(Q_{\mathrm{I,II}}/2) + (1/Q_{\mathrm{I,II}})]\delta^2(Z-Z'). \quad (6)$$

One can easily check that the quadratic algebras Eq. (5) satisfy Jacobi identities. The interesting point is that $M_{\mathrm{I,II}}$, which is now a dynamical quantity and has nonzero brackets with $\tilde{J}$, $\{M_{\mathrm{I,II}},\tilde{J}_{\mathrm{III}}\}_D = \tilde{J}_{\mathrm{III}}\check{M}_{\mathrm{I,II;III}}$, can be shown to satisfy modified Yang-Baxter relations, $[M_{\mathrm{I,II}},M_{\mathrm{III,I}}]+[M_{\mathrm{II,III}},M_{\mathrm{I,II}}]+[M_{\mathrm{III,I}},M_{\mathrm{II,III}}]+\check{M}_{\mathrm{I,II;III}}+\check{M}_{\mathrm{II,III;I}}+\check{M}_{\mathrm{III,I;II}}=0$, which we shall elaborate in a future publication when we quantize the system [15].

From the invariance of the action of J fields, Eq. (1), under the $V,\bar{V}$ transformations a conserved current equation can be derived, i.e., $\partial_y j = 0$, where $j(\bar{y},\bar{z}) \equiv 2\alpha\int_{-l/2}^{l/2} dz\, \tilde{J}\,\partial_{\bar{y}}\tilde{J}^{-1}$. Indeed this is also true in the canonical formulation, i.e., $\partial_y j = \{H_T, j\}_D = 0$. We then derive a set of Kac-Moody-like algebras,

$$\{j_{\mathrm{I}}(\bar{y},\bar{z}),j_{\mathrm{II}}(\bar{y}',\bar{z}')\}_D = 2\alpha l\mathbf{P}_{\mathrm{I,II}}\sum_n \delta'(\bar{y}-\bar{y}'+nL)\delta(\bar{z}-\bar{z}') + \mathbf{P}_{\mathrm{I,II}}(j_{\mathrm{II}}-j_{\mathrm{I}})\sum_n \delta(\bar{y}-\bar{y}'+nL)\delta(\bar{z}-\bar{z}'), \quad (7)$$

which can be easily shown to satisfy the Jacobi identity. The current $j(\bar{y},\bar{z})$ generates the left symmetry $\bar{V}(\bar{y},\bar{z})$ by the relations $\{\int_{-L/2}^{L/2} d\bar{y}\int d\bar{z}\,\mathrm{tr}[v(\bar{y},\bar{z})j(\bar{y},\bar{z})],\ \tilde{J}(\mathbf{x}')\}_D = v(\bar{y}',\bar{z}')\tilde{J}(\mathbf{x}')$, where $\bar{V}(\bar{y}',\bar{z}') \simeq 1 + v(\bar{y}',\bar{z}')$.

Similarly, other sets of quadratic algebras and their related Kac-Moody-like algebra of currents can be derived. Choosing $\bar{y}$ as time, and y as a space coordinate with periodic boundary condition, we obtain Eqs. (5) to (7) with the changes $J\leftrightarrow J^{-1}$, $\bar{y}\leftrightarrow y$, and $\bar{z}\leftrightarrow z$. Choosing z as time and $\bar{z}$ periodic, we obtain Eqs. (5) to (7), with the changes $y\leftrightarrow z$ and $\bar{y}\leftrightarrow\bar{z}$. Choosing $\bar{z}$ as time and z periodic, we obtain Eqs. (5) to (7) except with the changes $J\leftrightarrow J^{-1}$, $y\leftrightarrow\bar{z}$, and $\bar{y}\leftrightarrow z$. So, there are a total of four sets of algebras.

It is most interesting to note that the quadratic algebras Eq. (5) and the Kac-Moody algebras Eq. (7) of the SDYM system are the same in form as those of the WZNW model derived in Ref. [12] with the addition of two dimensions, which are ultralocal. This may give a hint that it is easier to go to higher-dimensional theories via the algebraic way. On the other hand, the two theories are very different in the following ways. (1) The Hamiltonian H_T of the SDYM system is interactive, in contrast to the Hamiltonian of the WZNW model, which is free. (2) The J fields cannot be factored into multiplication of "chiral" factors, i.e., $J \neq \mathcal{L}(\bar{y},\bar{z})\mathcal{R}(y,z)$, as in the WZNW model. (3) There are four distinct sets of algebras in the SDYM system, as compared to two in the WZNW model.

Virasoro-like algebras and hierarchies of nonlinear and linear systems.—Using the Sugawara construction [18], we can derive the Virasoro-like fields

$$U(\bar{y},\bar{z}) \equiv -(1/4\alpha L)\mathrm{tr}[j(\bar{y},\bar{z})]^2 = -(\alpha/L)\mathrm{tr}\left(\int dz\,\tilde{J}\,\partial_{\bar{y}}\tilde{J}^{-1}\right)^2$$

that satisfy the Virasoro-like algebras

$$\{U(\bar{y},\bar{z}),U(\bar{y}',\bar{z}')\}_D = \{2U(\bar{y},\bar{z})\delta'(\bar{y}-\bar{y}') + [\partial_{\bar{y}}U(\bar{y},\bar{z})]\delta(\bar{y}-\bar{y}')\}\delta(\bar{z}-\bar{z}'). \quad (8)$$

Note that these Virasoro-like algebras have no central charge. The Virasoro field $U(\bar{y},\bar{z})$ generates a nonlocal transformation on the field $\tilde{J}$,

$$\{U(\bar{y},\bar{z}),\tilde{J}(y',\mathbf{x}')\}_D = -L^{-1}\left(\int dz\,\tilde{J}\,\partial_{\bar{y}}\tilde{J}^{-1}\right)\tilde{J}(y',\mathbf{x}')\delta(\bar{y}-\bar{y}')\delta(\bar{z}-\bar{z}').$$

Therefore the fields $U(\bar{y},\bar{z})$ here have rather indirect relations with the conformal transformations. We do not expect these algebras to be as powerful as the Virasoro algebras in the WZNW model. The more important algebras are the quadratic algebras.

In the following, we derive hierarchies of nonlinear and linear systems of SDYM equations. It is known that an infinite number of involutive conservation quantities $I_n[U]$ can be constructed from the Virasoro-like field $U(\bar{y},\bar{z})$. The following bracket is important to consider, $\{j(\bar{y},\bar{z}),U(\bar{y}',\bar{z}')\}_D = \partial_{\bar{y}}(j(\bar{y},\bar{z})\delta(\bar{y}-\bar{y}')\delta(\bar{z}-\bar{z}'))$. The involutive conservation quantities take the following simple forms:

$$I_n[U] = n^{-1}\int d\bar{y}\,d\bar{z}[U(\bar{y},\bar{z})]^n,\quad n=1,2,\ldots;\quad \{I_n[U],I_m[U]\}_D = 0. \quad (9)$$

Now we can introduce new time variables t_n through the following equations:

$$\partial_{t_n} j(\bar{y},\bar{z},t_1,t_2,\ldots) = \{j(\bar{y},\bar{z},t_1,t_2,\ldots), I_n[U]\}_D = \partial_{\bar{y}}(jU^{n-1}),$$

where $n=1,2,\ldots$. These nonlinear equations form a hierarchy of nonlinear equations of the SDYM system. We will identify one of the times t_1 with $\bar{y}$. It is easy to write down the linear system,

$$\nabla_{t_n}\psi \equiv (\partial_{t_n} + jU^{n-1})\psi(\bar{y},\bar{z},t_2,t_3,\ldots) = 0 \qquad (n=1,2,\ldots), \tag{10}$$

such that the SDYM hierarchy nonlinear equations become the consequence of the curvatureless condition $[\nabla_{t_1},\nabla_{t_n}]=0$ $(n=2,3,4,\ldots)$. The other curvatureless conditions $[\nabla_{t_m},\nabla_{t_n}]=0$ $(m,n=2,3,4,\ldots)$ automatically follow. Therefore, the equivalence of the linear system to the hierarchy of nonlinear equations is proven. We can incorporate the conservation equations $\partial_y j=0$ into the SDYM hierarchy equations by adding one more equation, $\partial_y\psi(\bar{y},\bar{z},y,t_2,t_3,\ldots)=0$, to the linear system.

Similarly, three other sets of Virasoro fields and SDYM hierarchies of nonlinear equations can be constructed by choosing $\bar{y}$, z, and $\bar{z}$ as time, respectively.

This work is partially supported by the U.S. Department of Energy and the University of California INCOR program through the Center for Nonlinear Science at Los Alamos National Laboratory.

Note added.—After the distribution of this manuscript we saw a manuscript by Nair and Schiff [14] that gives a different canonical formulation of the SDYM equations. Their canonical formulation is for the Lie-algebra-valued gauge fields A_μ, in contrast to our group-valued gauge-invariant fields J; therefore they did not obtain the quadratic algebras which we consider to be more fundamental and important, especially for the quantization of the theory [15]. They also did not discuss the hierarchies of nonlinear and linear equations. The quantization of the SDYM has been done; see Ref. [15].

[1] A. A. Belavin, A. M. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. **59B**, 85 (1975).

[2] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)]; R. S. Ward, Commun. Math. Phys. **79**, 317 (1981).

[3] M. K. Prasad, A. Sinha, and L.-L. Chau Wang, Phys. Rev. D **23**, 2321 (1981). The $V(y,z),\bar{V}(\bar{y},\bar{z})$ symmetry of the J field equations was extensively discussed in this paper and was used for providing the reality of the multimonopole solutions.

[4] S. Donaldson, J. Diff. Geom. **18**, 279 (1983); **26**, 141 and 397 (1987); E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[5] A. A. Belavin and V. E. Zakharov, Phys. Lett. **73B**, 53 (1978); L.-L. Chau, in *Proceedings of the Guangzhou (Canton) Conference on Theoretical Particle Physics, 1980* (Van Nostrand Reinhold and Science, New York, 1981); K. Pohlmeyer, Commun. Math. Phys. **72**, 37 (1980).

[6] M. K. Prasad, A. Sinha, and L.-L. Chau Wang, Phys. Rev. Lett. **43**, 750 (1979).

[7] I. J. Mason and G. A. J. Sparling, Phys. Lett. A **137**, 29 (1989).

[8] R. S. Ward, Philos. Trans. R. Soc. London A **315**, 451 (1985).

[9] L. L. Chau and I. Yamanaka, University of California at Davis report (to be published).

[10] E. Witten, Phys. Lett. **77B**, 394 (1978); M. Sohnius, Nucl. Phys. **B136**, 461 (1978); I. V. Volovich, Phys. Lett. **129B**, 429 (1983); Theor. Math. Phys. **57**, 1269 (1983); C. Devchand, Nucl. Phys. **B238**, 333 (1984); L.-L. Chau, M.-L. Ge, and C.-S. Lim, Phys. Rev. D **33**, 1056 (1986); L.-L. Chau, in *Vertex Operators in Mathematics and Physics*, edited by J. Lepowsky, S. Mandelstam, and I. M. Singer (Springer-Verlag, Berlin, 1983).

[11] L.-L. Chau, Phys. Lett. B **202**, 238 (1988); L.-L. Chau and C. S. Lim, Phys. Rev. Lett. **56**, 294 (1986); L.-L. Chau, in *Integrable Systems*, Nankai Lectures on Mathematical Physics, 1987, edited by M. L. Ge and X.-C. Song (World Scientific, Singapore, 1989).

[12] L. L. Chau and I. Yamanaka, University of California at Davis Report No. UCDPHYS-PUB-91-5, 1991 (to be published).

[13] Such an action was first given by B.-Y. Hou and X.-S. Song, State University of New York at Stony Brook report, 1983 (unpublished), and recently by V. P. Nair and J. Schiff, Phys. Lett. **246B**, 423 (1990). [Other forms of action for SU(2) self-dual Yang-Mills equations in Yang's R gauge have also been given: K. Pohlmeyer, Commun. Math. Phys. **72**, 37 (1980); H. J. de Vega, Commun. Math. Phys. **116**, 659 (1988); however they are not useful for our considerations here.]

[14] V. P. Nair and J. Schiff, Columbia University Report No. CU-TP-521.

[15] L. L. Chau and I. Yamanaka, University of California at Davis Report No. UCDPHYS-PUB-91-19, 1991 (to be published).

[16] C. N. Yang, Phys. Rev. Lett. **38**, 1377 (1977); S. Ward, Phys. Lett. **61A**, 81 (1977); Y. Brihaye, D. B. Fairlie, J. Nuyts, and R. F. Yates, J. Math. Phys. **19**, 2528 (1978).

[17] L.-L. Chau, M. K. Prasad, and A. Sinha, Phys. Rev. D **24**, 1574 (1981); L.-L. Chau, in *Proceedings of the Conference on Nonlinear Phenomena, Oaxtepec, Mexico, 1982*, Lecture Notes in Physics Vol. 189, edited by K. B. Wolf (Springer, Berlin, 1982).

[18] H. Sugawara, Phys. Rev. **170**, 1659 (1968).