Suppression of the Rayleigh-Taylor Instability by Convection in Ablatively Accelerated Laser Targets

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An analytical model based on the WKB approximation is developed to describe the suppression of the short-wavelength Rayleigh-Taylor instability by convective flow. The boundary-value problem for the instability growth rate is reduced to the solution of a system of algebraic equations with the coefficients depending upon the unperturbed variables. The method, which is applicable for perturbation with wavelengths small compared with the density scale length, is applied to analyze the instability in stationary, ablating plasmas with convection and strong inhomogeneity.

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The Rayleigh-Taylor (RT) instability is the principal physical process inhibiting the achievement of sufficiently high compression of the fuel in inertial fusion. Recent experimental [1-4] and numerical results [1,5-8] indicate that over a wide range of conditions the growth of the RT instability is suppressed substantially by convective flow across the ablation region. The growth rate σ can be described by the so-called Takabe formula [6,7]:

$$\sigma = 0.9(gk)^{1/2} - 3kv\,, \qquad (1)$$

where k is the transverse wave number, v is the flow velocity across the ablation front, and g is the acceleration.

To gain further insight into the underlying physical process of the ablation stabilization, analytical solutions are of special importance. Such solutions might provide approximate scaling laws or solutions over a very large parameter space. Numerous attempts have been made to develop an analytical theory of the stabilizing effect of convection. For example, approximation of the problem by the discontinuity model has been made. This approach leaves the solution undetermined because it is necessary to introduce an unknown additional boundary condition to close the system of linearized equations for small perturbations [9-13].

For simplicity, let us consider here an incompressible model for the subsonic velocity of the ablation flow. We work in a frame of reference that moves with the center of mass of the plasma slab, which is accelerated by the ablation pressure.

It should be noted that the principal characteristic of this problem is the difficulty of approximating the problem with the discontinuity model. Indeed, an additional boundary condition for the unperturbed flow is needed for the discontinuity to be able to evolve [14]. But this condition specifies just the stationary-state discontinuity in terms of known variables. Therefore, the additional boundary condition, which is needed to pose the linearized problem, does not follow from a variation of an unperturbed additional condition. Moreover, we are not sure that it is possible to obtain this additional condition by any regular method. On the other hand, the property of evolution of a discontinuity is enough of a condition for the existence of a unique structure of the discontinuity [14]. It has been shown [8] that all the necessary information can be obtained from explicit consideration of the structure of the ablation region. That structure can be defined either by a one-dimensional numerical simulation or from an analytical self-similar solution. In this way the solution of the spectral problem can be obtained, at least numerically. Using this approach we do not have the extra variable that defines the position of the perturbed discontinuity surface. Therefore, no additional information is required in this approach.

It follows from a simple estimate and from numerical results [7,8] that suppression of the RT instability is substantial compared to the classical value for a perturbation wavelength $\lambda \simeq k^{-1} \simeq v/(g/L)^{1/2} \simeq 0.5$ μm. The perturbation wavelength is small compared to the density scale length $L \simeq 5-10$ μm in the unperturbed flow. Therefore, the "potential energy" is a slowly varying function of position, and the WKB approximation (in Russian books it is called the quasiclassical approximation) can be used with a high degree of accuracy [15].

Let us consider the stationary state of the unperturbed flow that is created during the irradiation of the target by the laser light. Typical profiles of the hydrodynamical variables in the laser plasma are given in Figs. 1-3 of Ref. [8] for different regimes of plasma irradiation and acceleration. The ablation region for these regimes is localized between the surface of peak density ($z = z_0$) and the surface where acceleration of the plasma particles vanishes ($z = z_a$). The acceleration and the density gradient are in the opposite direction; therefore, the condition necessary for RT instability occurs here.

The stability analysis proceeds by linearizing the set of unperturbed equations. For planar motion only two spatial directions need to be employed, and perturbations of the form $\Psi(t,x,z) = \psi(z)\exp(\sigma t + ikx)$ may be assumed. We assume the direction of the z axis to be along the gradients of the unperturbed flow. Furthermore, only growth rates σ that are large compared with the evolution of the unperturbed flow are considered. Assuming that the perturbations are adiabatic and the subsonic unperturbed flow is incompressible, we can write the linearized hydrodynamic equations for mass, momentum, and energy transport in the following forms:

$$\sigma\bar{\rho}+\frac{d}{dz}(\bar{\rho}v+\rho\bar{v}_z)+ik\rho\bar{v}_x=0\,, \tag{2}$$

$$\rho\left[\sigma\bar{v}_x+v\frac{d}{dz}\bar{v}_x\right]=-ik\bar{P}\,, \tag{3}$$

$$\rho\left[\sigma\bar{v}_z+\left[\frac{d}{dz}v\bar{v}_z\right]\right]+\bar{\rho}v\frac{dv}{dz}=-\frac{d}{dz}\bar{P}+\bar{\rho}g\,, \tag{4}$$

$$\frac{d}{dz}\bar{v}_z+ik\bar{v}_x=0\,, \tag{5}$$

where $\rho(z)$, $P(z)$, $v(z)$, and $g(z)$ are profiles of the unperturbed flow, and small perturbations from these profiles are labeled with an overbar.

The assumption of adiabaticity and incompressibility means that the following inequalities hold [12,13]:

$$\lambda/L \gg (l_e/LM)^{1/2} \quad (\lambda \ge 10^{-2}\,\mu\text{m})\,,$$

$$v \ll c_T,\quad k^{-1} \ll \max\{c_T^2/g, Lc_T^2/v^2\},\quad \sigma \ll kc_T\,,$$

where l_e is the mean electron free path, $M \ll 1$ is the Mach number, c_T is the isothermal sound velocity, and λ is the wavelength of the instability.

The growth rate of the instability is an eigenvalue of the boundary-value problem for the system of Eqs. (2)-(5) with the appropriate boundary conditions, which are characterized by a concrete physical problem.

We assume the following background structure of the hydrodynamic profiles. Let density, velocity, and temperature be constant outside the ablation region, i.e., downstream and upstream, and equal to the corresponding values at $z=z_a$ and z_0, respectively. The boundary conditions for Eqs. (2)-(5) are rapid evanescence of the perturbation modes away from the unstable region. Note that the eigenfunctions are localized around the unstable region in the WKB approximation; therefore, the solution is slightly dependent on the choice of the concrete model of the downstream and upstream flows.

To illustrate the WKB method let us consider first the growth of the RT instability for the plasma at rest. Then Eqs. (2)-(5) may be combined to obtain the known equation for $\bar{v}_z$ [16]:

$$\frac{d^2}{dz^2}\bar{v}_z-a(z)\frac{d}{dz}\bar{v}_z-k^2\left[1-\frac{ga(z)}{\sigma^2}\right]\bar{v}_z=0\,, \tag{6}$$

where $a(z)=-d\ln\rho/dz$ is the steepness of the density profile.

For perturbations with wavelength λ that is small compared to the scale length of the unperturbed flow, $L \simeq |a(z)|^{-1}$, we can omit the second term in Eq. (6). Without this term, Eq. (6) is analogous to Schrödinger's equation for the motion of a point mass particle with zero energy in the potential field $U(z)=k^2[1-ga(z)/\sigma^2]$. The eigenvalue σ_n for large values, $n \gg kL$, can be obtained using the Bohr-Sommerfeld quantization rule [15]:

$$k\int_{z_1}^{z_2}(ga/\sigma_n^2-1)dz=\pi(n+\tfrac{1}{2})\,, \tag{7}$$

where $n=1,2,3,\ldots$, and z_1,z_2 are the quasiclassical turning points. In particular, for $\rho(z)\propto\exp(-a_0 z)$ and $g=g_0=$const,

$$\sigma_n=\left[\frac{g_0 a_0}{1+[\pi(n+1/2)]^2/(kL)^2}\right]^{1/2},$$

which for $n \gg kL$ coincides with the exact solution [8,16].

Note that although the exponential profile for the plasma at rest has been solved analytically [16], using the WKB approach we obtain the eigenvalue spectrum for an arbitrary density profile in the explicit analytical form.

Let us now consider the suppression of the short-wavelength RT instability by plasma convection across the ablative region. For the WKB approximation the eigenfunctions of Eqs. (2)-(5) may be written in the form

$$\Psi(z)=\sum_j A_j\exp\left[k\int^z\varphi_j(z')dz'\right]\,, \tag{8}$$

where $\Psi=\bar{v}_z,\bar{v}_x,\bar{\rho},\bar{P}$. Equations (2)-(5) together with Eq. (8) yield a homogeneous algebraic system in the amplitudes A_j. The phase functions $\varphi_j(z)$ are the roots of the fourth-order characteristic equation obtained by setting the determinant of this system equal to zero:

$$\mathcal{R}(\varphi)\equiv(\Sigma+\varphi)^2(\varphi^2-1)-(\Sigma+\varphi)\frac{gM^2}{v^2k^2}\frac{d\ln v}{dz}\varphi+\frac{1}{4}G$$
$$=0\,, \tag{9}$$

where we have used dimensionless variables, growth rate $\Sigma(z)=\sigma/vk$, and acceleration

$$G(z)=4ga(z)/(vk)^2=4a^2/k^2\mathrm{Fr}\,,$$

with $\mathrm{Fr}=v^2a/g$ the Froude number. The term $\bar{\rho}v\,dv/dz$ in Eq. (4), small compared to $\bar{\rho}g$, is neglected.

For subsonic flow the second term in Eq. (9) is small, since $M^2 \ll 1$, and can be omitted. The location of the roots φ_j of Eq. (9) in the complex plane φ depends on the profile coordinate z. At infinity, where the flows are assumed to be uniform, the roots φ_j correspond to sonic, entropy, and vorticity modes [8,12]:

$$\varphi_1(z\to\mp\infty)=1,\quad \varphi_2(z\to\mp\infty)=-1\,,$$

$$\varphi_{3,4}(z\to\mp\infty)=-\Sigma\,.$$

Not more than two roots can be simultaneously located in the half plane $\mathrm{Re}(\varphi)>0$. At certain values of z the roots can intersect and form multiple ones. We expect perturbations, which are localized in the ablative region, to have the following asymptotic behavior at infinity:

$$\Psi(z)=A\exp(kz\varphi_1) \text{ for } z\to-\infty\,, \tag{10}$$

$$\Psi(z)=\sum_{j=2}^{4}A_j\exp(kz\varphi_j) \text{ for } z\to+\infty\,. \tag{11}$$

Such asymptotic behavior is possible if points along the unperturbed profile exist where intersection of the roots of Eq. (9) occurs in the half plane $\mathrm{Re}(\varphi)>0$. Thus, the condition that perturbations are localized in the unstable region can be written as

$$\mathcal{R}(\varphi)=0 \text{ and } d\mathcal{R}(\varphi)/d\varphi=0\,. \tag{12}$$

Denoting the real solution of Eqs. (12) by φ_* and Σ_*, and $s=(1+G/27)^{1/2}$, we have

$$\Sigma_*=(1-2\varphi_*^2)/\varphi_*\,, \tag{13}$$

$$\varphi_*=\left[1-\frac{G^{1/3}}{(s+1)^{2/3}+(s-1)^{2/3}+\frac{1}{3}G^{1/3}}\right]^{1/2}. \tag{14}$$

The maximum growth rate σ can be obtained from the system of algebraic equations

$$\Sigma(z)=\Sigma_*(z),\quad \frac{d}{dz}[\Sigma(z)]=\frac{d}{dz}[\Sigma_*(z)]\,. \tag{15}$$

We do not consider the remaining conjugate complex solutions of Eqs. (12) which could be associated with complex eigenvalues σ. Equations (15) are overdetermined in this case and do not have a solution in a common case.

Using the explicit form of the functions $\Sigma(z)$ and $\Sigma_*(z)$, we can eliminate σ to obtain the equation that defines the only root intersection point $z=z_*$:

$$\frac{d}{dz}\left[\frac{g}{v^2k^2}\frac{d}{dz}\ln\rho\right]=\frac{1}{2}\left[\frac{1-2\varphi_*^2}{1+2\varphi_*^2}\right]\times[12(1-\varphi_*^2)^2+G]\frac{d}{dz}\ln v\,, \tag{16}$$

where $\varphi_*=\varphi_*(G(z))$ is defined by Eq. (14).

Thus, we obtain the solution for the maximum growth rate,

$$\sigma=kv(z_*)\Sigma_*(z_*)\,. \tag{17}$$

Equations (16) and (17) are the required WKB solution to the problem.

The WKB approach allows us in principal to find the spectrum σ_n if the behavior of the roots of Eq. (9) in the complex plane φ is known.

The eigenfunction that corresponds to the maximum eigenvalue is

$$\Psi(z)=\exp\left[k\int_{z_*}^{z}\varphi_+(z')dz'\right],\quad z>z_*\,,$$
$$\Psi(z)=\exp\left[k\int_{z_*}^{z}\varphi_-(z')dz'\right],\quad z<z_*\,, \tag{18}$$

where φ_+ is the root of Eq. (9) which shifts along the axis to $\mathrm{Im}(\varphi)=0$ at the half plane $\mathrm{Re}(\varphi)<0$ when $z>z_*$, while φ_- is the root which tends to the point $\varphi=1$ when $z<z_*$. Depending on the eigenvalue σ, the root φ_+ represents either the vorticity or the sonic mode in the downstream region. An analysis has revealed that the former case is realized if the condition $\sigma<kv(+\infty)$ is valid, while the latter is realized in the opposite limit.

The function $\Sigma_*(G(z))$ reaches its maximum value at the point where $G=G_{\max}[-4g(vk)^{-2}d\ln\rho/dz]$. This means that $\Sigma_*(G(z))$ is negative if $G_{\max}<1$, and the growth rate vanishes if $G_{\max}=1$. From this condition we can find the value of the wave number k_0 for which the growth rate vanishes:

$$k_0=[\max(-4gv^{-2}d\ln\rho/dz)]^{1/2}\,. \tag{19}$$

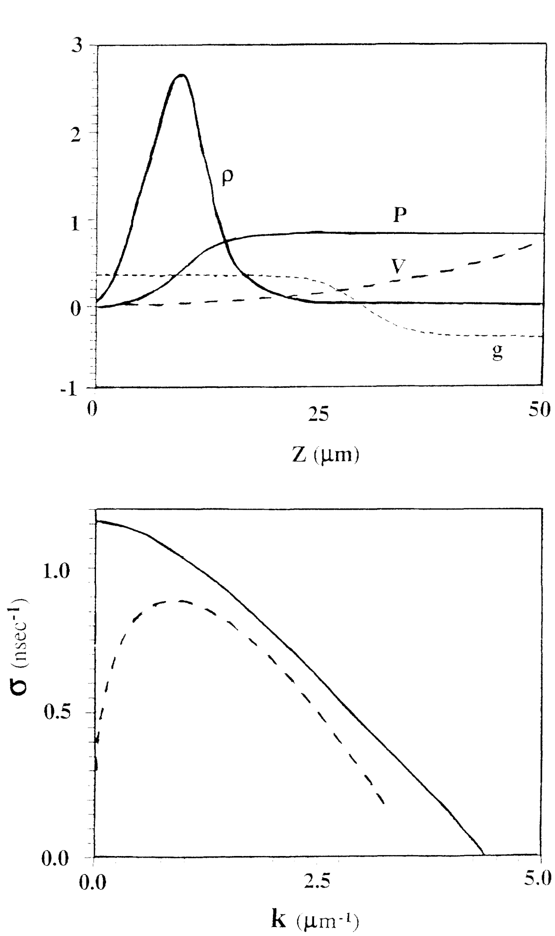

FIG. 1. Top: Steady-state smoothed unperturbed profiles of density ρ (g/cm^3), pressure P (Mbar), flow velocity V (10^6 cm/s), and acceleration g (10^{15} cm/s^2) in the ablation plasma flow for a 10-μm Al planar target exposed to Nd:glass laser radiation ($\lambda_0=1.05$ μm, $I=10^{13}$ W/cm^2). Bottom: The RT growth rates of instability corresponding to the profiles in the top figure vs wave number k obtained numerically in Ref. [8] (dashed line) and from the WKB approach (solid line).

It is clear that, strictly speaking, we can consider vanishing of the growth rate for the stationary state of the unperturbed motion only. Note that although formula (19) reveals a scaling for k_0 quite different from the approximation of Takabe's formula (1), the numerical values of k_0 are close in both cases for parameters typical of ablating plasmas. The discrepancy has to be appreciable for smoother unperturbed profiles.

Figure 1 demonstrates spectra of the RT growth rates in an aluminum target of thickness 10 μm exposed to Nd:glass laser radiation ($\lambda_0 = 1.05$ μm, irradiation $I = 10^{13}$ W/cm^2). The smoothed unperturbed plasma profiles are shown at the top in the coordinate system comoving with the center of mass of the accelerated target. The dashed line in the bottom panel represents the disperson curve obtained numerically by the shooting method as an eigenvalue of the boundary-value problem; the solid line corresponds to the WKB solutions where the unperturbed hydrodynamic profiles (the top profiles) are used as the coefficients in Eqs. (16) and (17). The difference between the numerical and WKB solutions depends on the applicability of the WKB approach, which implies that the local Froude number must be small compared to unity. In the numerical unperturbed solutions shown in the upper figure, $\mathrm{Fr} = v^2 a/g \simeq 0.25$.

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[1] J. D. Kilkenny, Phys. Fluids B **2**, 1400 (1990).
[2] R. L. McCrory *et al.*, Plasma Phys. Controlled Fusion **31**, 1517 (1989).
[3] J. Grun *et al.*, Phys. Rev. Lett. **58**, 2672 (1987).
[4] M. H. Emery, J. H. Gardner, and S. E. Bodner, Phys. Rev. Lett. **57**, 703 (1986).
[5] J. H. Gardner, S. E. Bodner, and J. P. Dahlburg, Phys. Fluids B **3**, 1070 (1991).
[6] H. Takabe, L. Montierth, and R. L. Morse, Phys. Fluids **26**, 2299 (1983).
[7] H. Takabe, K. Mima, L. Montierth, and R. L. Morse, Phys. Fluids **28**, 3676 (1985).
[8] V. V. Bychkov, S. M. Goldberg, and M. A. Liberman, Zh. Eksp. Teor. Fiz. **73**, 642 (1991) [Sov. Phys. JETP **100**, 1162 (1991)].
[9] S. E. Bodner, Phys. Rev. Lett. **33**, 761 (1974).
[10] J. R. Catto, Phys. Fluids **21**, 30 (1978).
[11] L. Baker, Phys. Fluids **21**, 295 (1978).
[12] W. H. Manheimer and D. G. Colombant, Phys. Fluids **27**, 983 (1984).
[13] H. J. Kull and S. I. Anisimov, Phys. Fluids **29**, 2067 (1986).
[14] M. A. Liberman and A. L. Velikovich, *Physics of Shock Waves in Gases and Plasmas* (Springer-Verlag, Berlin, 1986).
[15] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977).
[16] K. O. Mikaelian, Phys. Rev. A **26**, 2140 (1982).