Observations of Localized Structures in Nonlinear Lattices: Domain Walls and Kinks

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(Received 22 October 1991)

Steady-state domain walls and kinks have been observed in one-dimensional nonlinear lattices that are damped and parametrically driven. These states are localized robust transition regions between two extended standing-wave domains of definite wave number. The observations are made in an experimental lattice of coupled pendulums and in simplified numerical models. A nonlinear Schrödinger theory is developed for kinks in the upper cutoff mode. There is currently no theory for the domain walls and noncutoff kinks, which are fundamentally new localized structures.

PACS numbers: 63.20.Pw, 46.10.+z, 63.20.Ry

We have found that nonlinear vibratory lattices can have a wealth of nonpropagating self-localized structures [1]. Here we report the observation of *domain walls* which connect standing-wave regions of different wave number (Fig. 1), and *kinks* which connect standing-wave regions of the same wave number with a spatial phase mismatch (Figs. 2 and 3). The robust nature of these states is evident in the kinks of Fig. 2, which are smooth envelopes of a mode that is nonuniform due to lattice irregularities. As with the nonlinear Schrödinger (NLS), Korteweg-de Vries (KdV), sine-Gordon, and Toda solitons, these localized structures represent a spontaneous breaking of the translational invariance that characterizes the underlying equations of motion. In addition, the domain wall breaks parity. We have found that kinks in the upper cutoff mode (in which each oscillator is 180° out of phase with its immediate neighbors) are approximately described by an NLS equation, whereas domain walls and noncutoff kinks are not described by this or the KdV, sine-Gordon, or Toda equations. A theoretical understanding of the new structures may lead to new generic integrable equations, and thus to fundamentally new solitons.

The observations of domain walls and kinks in an actual lattice and various simplified numerical models indicate that these structures are general phenomena which can occur in many other lattices. For example, upper cutoff kinks should be observable in the motion of the separatrices in a linear array of vortices [2]. Furthermore, domain walls and noncutoff kinks may exist as nonpropagating or propagating structures in continua, as is the case with lower cutoff [3] and similar [4] kinks. The considerations in this article are restricted to one space dimension. However, the robust nature of our observations suggests searching for generalizations of these phenomena in higher dimensions.

An underlying theme here is the high degree of complexity of the modal structure of nonlinear mesoscopic systems. For example, there exist more physically distinct modes than are enumerated by the degrees of freedom of the system. One can gain appreciation of this by our observation of a bound state of a domain wall and a

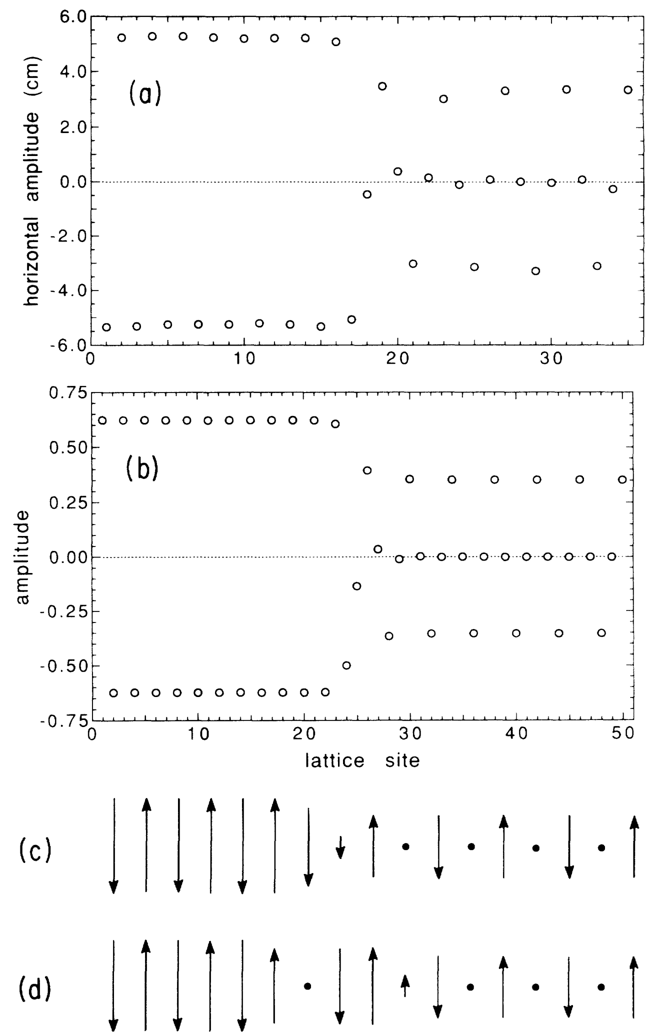

FIG. 1. Domain wall between the upper cutoff and wavelength-four modes: (a) experimental horizontal amplitudes of the oscillators; (b) numerical amplitudes; and diagrammatic representations of peak-to-peak amplitudes and phases of a highly localized (c) domain wall and of a (d) bound state of a domain wall and kink. In (a), the drive frequency is 3.68 Hz and the peak drive amplitude is 0.52 mm. In (b), the dimensionless drive parameters are $\omega/\omega_0 = 1.07$ and $\eta/\omega_0^2 = 0.10$.

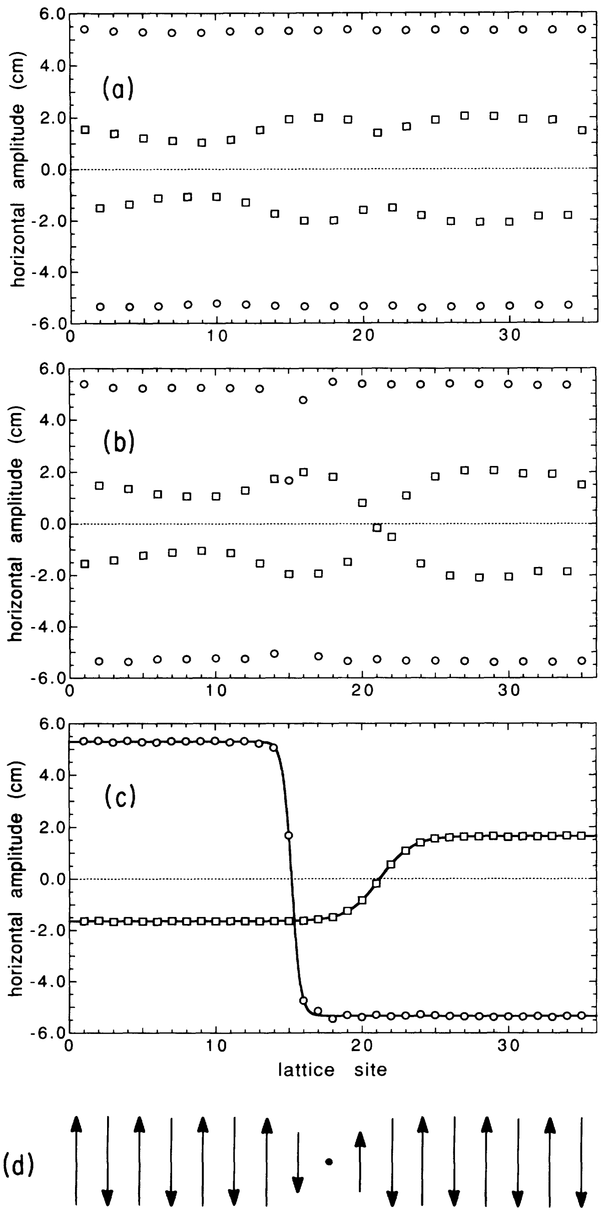

FIG. 2. Experimental data of (a) the upper cutoff mode, (b) kinks in this mode, (c) the values of the latter divided by those of the former and then scaled to the original average value, and (d) diagrammatic representation of a highly localized antisymmetric kink in a uniform lattice. The curves are hyperbolic tangent best fits. The drive frequency is 3.95 Hz for the smaller amplitude data, and 3.68 Hz for the larger amplitude data. The peak drive amplitude is 0.80 mm in both cases.

kink [Fig. 1(d)].

The apparatus is remarkably simple. It is a lattice of 35 pendulums supported by a rod that is attached to a vertical-shake table (Fig. 4). Each pendulum consists of strong thread in a shape similar to a V with a brass cylindrical bob at the bottom. The V's are constructed to overlap, where knots are tied to ensure the coupling. The

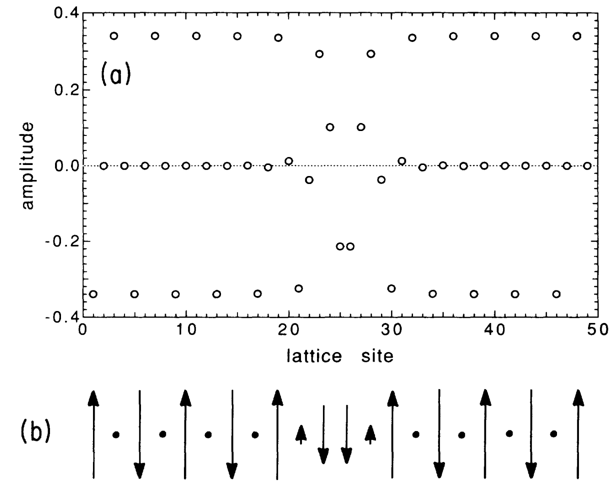

FIG. 3. Symmetric kink in the wavelength-four mode: (a) numerical amplitudes and (b) diagrammatic representation of the highly localized state. In (a), the dimensionless drive parameters are $\omega/\omega_0 = 1.07$ and $\eta/\omega_0^2 = 0.09$.

phenomena reported here have also been observed in a similar lattice made simply of kite string and machine nuts. To minimize end effects, we clamp each outer segment of thread at the location appropriate for the mode of interest. Although the range of amplitudes in the apparatus is limited compared to sine-Gordon lattices [5], this has no essential effect upon the localized structures reported here. For large-amplitude motion in our lattice, the pendulum bobs twist substantially. That the localized structures continue to exist under this condition is an indication of their robustness.

One means of obtaining a domain wall in the lattice is to begin with a pure upper cutoff mode by driving at approximately twice [6] the linear frequency of the mode. Next, lower the drive frequency to a value near twice the linear frequency of the wavelength-four mode, and then eliminate the motion in part of the lattice by touching the bobs with a meter stick. The wavelength-four mode will then develop and eventually reach a steady-state coexistence with the upper cutoff mode. Steady-state amplitude data are obtained from digitized videotape. The

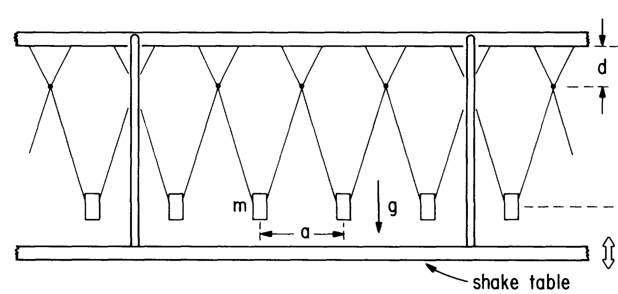

FIG. 4. Pendulum lattice apparatus. The motion is approximately perpendicular to the lattice. The values of the parameters are $L = 8.4$ cm, $d = 1.9$ cm, $a = 2.5$ cm, and $m = 13$ g.

video camera is directed horizontally at a mirror inclined at 45° and located between the shake table and lattice. Figure 1(a) shows the displacements of the pendulums near a turning point of the motion. We have also observed domain walls between other modes.

To gain insight into the generality of the phenomena, we have carried out numerical studies of a highly idealized version of the actual lattice. In this model the motion of each mass is one dimensional, the coupling purely linear, and the nonlinearity only cubic. None of these assumptions holds for the actual lattice. With the inclusion of linear damping and parametric drive of the external linear potential wells, the equation of motion of the model system is

$$\ddot{\theta}_n - c^2(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + \beta\dot{\theta}_n + [\omega_0^2 + \eta\cos(2\omega t)]\theta_n = \alpha\theta_n^3, \quad (1)$$

where dots denote time differentiation, θ_n is the displacement of the nth oscillator, ω_0 is the linear frequency of an uncoupled oscillator, η is the drive amplitude, 2ω is the drive frequency, β is the damping parameter, and c^2 is a measure of the coupling strength. The nonlinear coefficient α is $\omega_0^2/6$ if (1) is to approximate a pendulum lattice. We choose a cubic nonlinearity because it is the simplest, and because we wish to show that the $\sin(\theta)$, which occurs in the equation of motion of the actual lattice, plays no essential role in the existence of the localized states. Furthermore, the cubic nonlinearity is advantageous because we can generalize the numerical model by allowing the coefficient α to be positive (softening system), zero (linear system), or negative (hardening system). By "softening" or "hardening" is meant that the frequency of a free-standing wave decreases or increases, respectively, at greater amplitudes. By scaling time and displacement, we can normalize ω_0 and the magnitude of α in (1). Unless otherwise specified, the results presented here are for a softening system ($\alpha = +1$) with $c^2 = 0.1$ and $\beta = 0.03$. The value of c^2 roughly equals the actual value [7] in the experimental lattice, while the value of β is an order of magnitude greater than the actual value. This greater value was chosen to more quickly eliminate transients, and has no essential effect upon the localized states. We find that periodic boundary conditions yield essentially the same results as "reflecting" boundary conditions, in which ghost oscillators next to the ends of the lattice are specified to have the same instantaneous displacements as the oscillators one wavelength from the ghost oscillators. Our domain-wall investigations were performed with these boundary conditions, because in this case a single domain wall can occur in the lattice. If the number of oscillators does not exceed several hundred, the numerical solution of (1) can be handled on a fast personal computer with a simple finite-difference method (e.g., fourth-order Runge-Kutta method). The stability of all states was checked by imposing random perturbations of the displacements near a turning point. Numerical domain walls [Fig. 1(b)] are very similar to experimental ones, and have also been observed between a variety of modes. Furthermore, we have numerically simulated domain walls in (1) with a hardening nonlinearity ($\alpha = -1$) and with a quadratic rather than a cubic nonlinearity, as well as in damped parametrically driven lattice generalizations of the sine-Gordon and sinh-Gordon equations.

Of the localized structures reported here, the easiest to observe experimentally are upper cutoff kinks (Fig. 2), which are localized transition regions between two upper cutoff domains that are mismatched by 180°. If the pendulums are initially at rest and the system is driven at twice the linear upper cutoff frequency, a pure upper cutoff mode seldom develops. Instead, one or more kinks occur. When two kinks are brought near each other by hand, they attract and eventually annihilate, in contrast to the surface-wave case in which there is strong repulsion [3]. We have numerically observed antisymmetric [Fig. 2(d)] and symmetric kinks [8]. Typical experimental upper cutoff mode and kink data are shown in Figs. 2(a) and 2(b), respectively. The kink locations are different because lattice nonuniformities caused the kink to migrate as the response amplitude was increased. The uncertainty of the measurements is negligible (±0.3 mm); the lack of smoothness of the data is a result of nonuniformities of the lattice. The low amplitude data in Fig. 2(a) show a ±30% peak variation, even though the peak nonuniformity of the lattice dimensions is only ±3%. The large enhancement of the fractional variation in the response is suggestive of a standing-wave analog of Anderson localization [9]. That the large amplitude data in Fig. 2(a) show only a small amount of variation is indicative of strong nonlinearity restoring the translational invariance broken by Anderson localization [10]. Figure 2(c) shows the results of dividing the kink data by the upper cutoff mode data, and then scaling the amplitudes to the average of the original values. The data are smoothed by this process, and are fitted remarkably well by a hyperbolic tangent function. Numerical studies show that the width of this function is not independent of the location of the kink. The hyperbolic tangent is a characteristic solution of the nonlinear Schrödinger (NLS) theory for a uniform (translationally invariant) lattice, as we now show.

We begin with (1) and consider an amplitude modulation of the upper cutoff mode:

$$\theta_n(t) = (-1)^n \mathcal{A}(n,t) e^{i\omega t} + \text{c.c.} + \cdots , \quad (2)$$

where $\mathcal{A}(x,t)$ is a continuous differentiable complex function of its arguments, ω deviates a small amount from the linear frequency $\omega_1 = (\omega_0^2 + 4c^2)^{1/2}$ of the upper cutoff mode, the lattice spacing is unity, and the ellipsis denotes higher harmonics. When (2) is substituted into (1), and $\mathcal{A}$ is assumed to be weakly nonlinear and slowly varying in space and time, it can be shown that $\mathcal{A}$ obeys an NLS equation if the drive and dissipation are also weak [7]:

$$2i\omega\mathcal{A}_t+c^2\mathcal{A}_{xx}+(\omega_1^2-\omega^2-i\omega\beta)\mathcal{A}-\eta\mathcal{A}^*/2 =3\alpha|\mathcal{A}|^2\mathcal{A}. \quad (3)$$

This equation is ubiquitous in the description of modulations of damped parametrically driven cutoff modes. It arises, for example, in a mass-and-spring lattice with nonlinear springs. The equation is integrable in the case of no drive and dissipation [11]. We consider steady-state solutions of the form $\mathcal{A}(x,t)=A(x)e^{i\delta}$, where A and δ are real. The stable single-kink solution of (3), which exists for a softening nonlinearity ($\alpha>0$), is [3]

$$A(x)=(\gamma/3\alpha)^{1/2}\tanh[(\gamma/2)^{1/2}(x-x_0)/c], \quad (4)$$

and $\tan(2\delta)=\omega\beta/\nu$, where $\gamma=\omega_1^2-\omega^2+\nu$, $\nu=(\eta^2/4-\omega^2\beta^2)^{1/2}$, and the location x_0 of the node is arbitrary. The approximate theoretical expression (4) agrees very well with numerical antisymmetric kink data based on (1), even when the amplitudes appear to be neither weakly nonlinear nor slowly varying. For example, the deviation is only about 1% when $\theta_n=0.80$ in the uniform region and $\Delta\theta_n/\Delta n=0.75$ in the kink region.

Kinks are not restricted to cutoff modes, but can also exist in modes in the band. Figure 3(a) shows numerical data of a symmetric kink in the wavelength-four mode. This kink has also been observed in the actual pendulum lattice. There is a 90° spatial phase mismatch between the domains on either side of the kink. The mismatch is easily identified when the kink is highly localized [Fig. 3(b)]. We have also observed an antisymmetric kink corresponding to this mismatch, as well as symmetric and antisymmetric kinks corresponding to the two other possible mismatches (−90° and 180°) [8]. Noncutoff kinks have been observed in a variety of modes, and have been numerically simulated in the lattices mentioned above. In fact, our experience with the many different numerical models leads us to claim that all of the structures reported here can exist in any lattice whose description has the form of (1) with any nonlinear potential, provided that the associated extended mode (or modes) is stable. This disallows hardening upper cutoff and softening lower cutoff modes, which exhibit the Benjamin-Feir instability and evolve into localized breather states [7].

As in the case of domain walls, there currently exists no theory for noncutoff standing-wave kinks. The above analytical approach, which is successful in the case of cutoff modes, is not sufficiently general to describe the structures in noncutoff modes. Finally, we reiterate our observation that upper cutoff kinks are smooth envelopes of eigenmodes that can be highly nonuniform. That these envelopes have the same functional form as solutions to the NLS equation for a translationally invariant lattice is also not understood.

We are grateful to R. Keolian, who was primarily responsible for the basic design of the apparatus. This work was supported in part by a joint fellowship from the Office of Naval Technology and the American Society for Engineering Education, by the Office of Naval Research, and by the Division of Engineering and Geophysics, Office of Basic Energy Science, Department of Energy.

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