Deuteron in the Skyrme Model

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(Received 13 November 1991)

Classical bound-state solutions for the baryon number =2 sector of the Skyrme model have been found by numerical simulation in 3+1 dimensions. The Bohr-Sommerfeld level of quantization is implemented. Properties of the deuteron obtained by quantization about the periodic classical solution improve on previous work based on quantization about the minimum energy static toroidal configuration.

PACS numbers: 21.45.+v, 11.15.Kc, 11.40.Fy, 27.10.+h

QCD has had little direct impact on low-energy nuclear physics. While the high-energy behavior of QCD can be simply understood in terms of quarks and gluons, at low energies their dynamics is extremely nonperturbative. However, a possible framework for low energy follows from a dual description of QCD that is suggested by the $1/N$ expansion. In this conceptualization of QCD, the baryons and nuclei arise as topological solitons [1] of a stringlike theory of meson dynamics [2]. Such a description of QCD, exact in principle, could lead to a simple fundamental description of low-energy nuclear physics if $1/N$ is effectively a small expansion parameter, even for $N=3$.

Until the string theory of meson dynamics is determined, one can begin to explore this framework with the Skyrme model [3]. This model is an effective chiral Lagrangian [4] for low-energy QCD,

$$L_{\rm QCD}\to L_{\rm eff}=\frac{f_\pi^2}{16}{\rm tr}\,\partial_\mu U\,\partial^\mu U^{-1}+\frac{1}{32e^2}{\rm tr}[\partial_\mu U U^{-1},\partial_\nu U U^{-1}]^2+\cdots,$$

where $U(\mathbf{x},t)=\exp[2i\pi^a(\mathbf{x},t)\lambda^a/f_\pi]$; $L_{\rm eff}$ describes QCD with massless quarks for all wavelengths $>\hbar/m_\rho c\sim\frac{1}{4}$ fm. It results, in a renormalization-group sense, from integrating out all quantum fluctuations with wavelengths smaller than $\hbar/m_\rho c$ [5]. Although U should be an element of SU(3), in the following, we will take it to be an SU(2) field. In QCD, the $1/N$ perturbation theory gives an ordering of importance of physical effects. If $f_\pi\sim\sqrt{N}$, and $e\sim 1/\sqrt{N}$, then $L_{\rm eff}$ satisfies all QCD $1/N$ perturbation theory counting rules. Remarkably, this effective Lagrangian of low-energy pion dynamics [4] also has nucleons [6], baryon resonances [7], and nuclei [8] as topological soliton excitations. If $1/N$ is effectively a small parameter for QCD, then the solitons are semiclassical.

If nuclei are semiclassical bound states of solitons, then their spectrum can be studied in the following way [9]. In general, quantum bound states are described by the poles of

$$\mathrm{tr}\frac{1}{E-H}=-i\int_0^\infty dT\,e^{iET}\,\mathrm{tr}\,e^{-iHT},$$

where

$$\mathrm{tr}\exp\left(\frac{i}{\hbar}HT\right)=\int_{U(\mathbf{x},t+T)=U(\mathbf{x},t)}DU(\mathbf{x},t)\times\exp\left(\frac{i}{\hbar}S[U(\mathbf{x},t)]\right),$$

and the trace of the time evolution operator is expressed in terms of a functional integral over field configurations that are periodic in time T. The semiclassical approximation is dominated by field configurations that satisfy $\delta S=0$, the stationary phase approximation. We are therefore interested in studying periodic field configurations (of given B) that satisfy the classical equations of motion.

We will work in rescaled units: E is measured in units of $f_\pi/e\sim N$, $\mathbf{x}$ and t are in units of $1/ef_\pi\sim 1$, and, consequently, in these rescaled units the action S is proportional to $1/e^2\sim 1/(1/N)$, showing that $1/N$ plays the role of $\hbar$.

Since we are interested in the $B=2$ bound states, we are led to study energetically bound, periodic solutions. We will examine the maximally attractive channel [10,11] in which static $B=2$ solutions have previously been found [12–15]. Our approach is to use numerical finite difference techniques to simulate the classical evolution of the Skyrme model and search for periodic solutions. This entails certain approximations. Our computational domain is a box in which the field U approaches the unit matrix at the edges of the box. In the experiments described below, the box is 16 by 16 by 8 (in scaled units). By use of symmetry planes, the actual computational domain is reduced by a factor of 8. The reduced finite difference grid has $56\times56\times28$ lattice points, which leads to a lattice spacing of 0.143. The finite difference scheme [16] used a Courant-Friedrich-Levy ratio ($\Delta T/\Delta X$) of 0.5. This improvement over previous finite difference schemes [17] allows us to study low-energy interactions necessary for the bound-state problem.

Initial field configurations were prepared consisting of two separated Skyrmions. The continuum Skyrmion field configuration is not a solution on the finite lattice. Instead, the initial configurations were prepared by relaxation. The continuum Skyrmion solution is imposed upon the lattice, then evolved forward in time with a viscous attenuation. The initial separation was eight scaled units. As the Skyrmions relax, they approach each other as their orbital energy is dissipated. By stopping the relaxation procedure after varying numbers of relaxation steps, we produce a family of initial conditions characterized by decreasing total energy and diminishing distance between the Skyrmions. The initial separation [18] varies smoothly with energy, from 3.15 for the lowest-energy run of 71.99 near the toroidal minimum to 7.23 for the highest-energy run of 74.40 near the $2M_{\text{Skyrmion}}$ continuum. In other words, we are examining a range in maximum separations of approximately 1.5 to 3.5 times the radius of a single Skyrmion.

For each member of the family of initial conditions, we evolve the solution forward in time without attenuation. At the point of closest approach, the two solitons merge into a toroidal field configuration similar to the static $B=2$ solution [13–15]. The Skyrmions then separate at right angles [11]. The process repeats with the Skyrmions falling back toward each other, forming a toroidal configuration, then scattering at right angles. When the Skyrmions reach their point of maximum separation after the second pass, they are very near to their starting configuration. The configuration is not identical to the initial configuration since, during the large field deformations at closest approach, energy was transferred from the orbital motion into an internal radial excitation of each Skyrmion [19] (see Fig. 1). Since the period of the internal oscillation is much smaller than the orbital period, we will ignore the internal excitation in the following and treat our nearly periodic solution as representative of the true periodic solution.

The Bohr-Sommerfeld quantization requires that we measure the energy, action, and period of the periodic solution. In order to avoid the ambiguities of measurement created by the transfer of orbital energy to internal excitations, we measure the action and period in the first quarter of the orbit as the Skyrmions fall together. We determine the quarter period from the minimum of the kinetic to the minimum of the potential energies. We multiply these values by 4 to represent the action and period of the true periodic solution. The finite difference method conserves energy to 0.5% or better. We take the energy to be the average of measured energy over an orbital period. As a consistency check, we have also computed the energy from the relation $E=-dS/dT$. The two methods of measuring the energy agree to within 0.2%. The agreement suggests that our methods of determining the period, action, and energy are consistent.

From the analysis of Dashen, Hasslacher, and Neveu [9] we have

$$\operatorname{tr}\frac{1}{E-H}\approx\frac{1}{1+e^{iW(E)/e^2}},$$

where

$$W(E)=S(T(E))+ET(E)+\text{quantum corrections},$$

with S the classical action for the period T. The plus sign in the denominator is due to the existence of two turning points in the orbit. The bound-state poles arise from the condition

$$W(E)=(2n+1)\pi e^2,$$

where again E is measured in units of f_π/e. In Fig. 2 we present our results for W vs E on the finite lattice. The curve extends from energies corresponding to the minimum of the potential well to the continuum of two noninteracting Skyrmions. (Because of finite lattice effects, the value $2M_{\text{Skyrmion}}\sim 74.6$ is larger than the continuum result ~ 73. Until we have repeated our experiments on larger lattices and extrapolated to the continuum limit, our results are largely qualitative.) The number of bound states predicted by the quantization condition is determined by the value of e^2. For e^2 large (the continuum value has been estimated in the $B=1$ sector to be approximately 30), there is only a single bound state. In nature, the $B=2$ system has only a single weakly bound state, the deuteron. Quantization about the static configuration [13–15] would only be a good approximation if e^2 were small, in which case there would be many $B=2$ bound

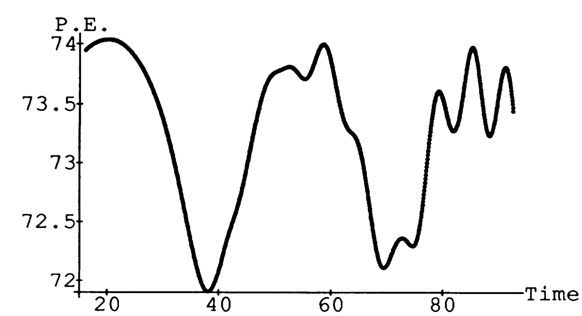

FIG. 1. Potential energy as a function of time for $E=74.0$. Orbit starts at $t=20.2$ with Skyrmions at greatest separation. At $t=38.1$, Skyrmions have merged into toroidal configuration. At $t\sim 56$, Skyrmions are well separated, but relative axis has rotated by 90° from initial configuration. Note transfer of orbital energy into internal excitation with short period.

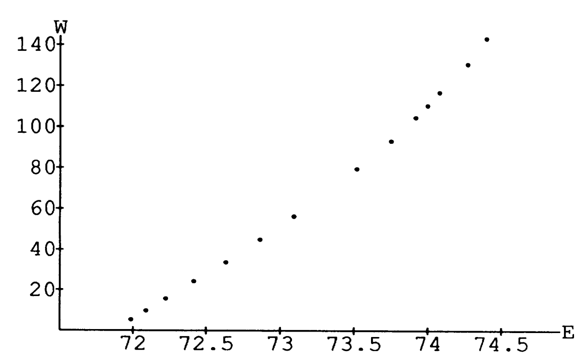

FIG. 2. Quantization function $W(E)$ that determines the semiclassical bound-state spectrum from the condition $W=(2n+1)\pi e^2$.

states.

For large e^2, our semiclassical picture of the $B=2$ system has a large quadrupole moment and small energy splittings between spin-isospin states. Since the dynamical solution for large e^2 spends most of its time with the Skyrmions well separated, the radius of the system is about 2 times that of the static solution. Consequently, the quadrupole moment should be about 4 times larger than for the static solution. The quadrupole moment of the static solution [13] is about $\frac{1}{4}$ of the experimental value. The quantization of the global rotation and isospin collective coordinates gives small splittings for the fine structure of states. Expanding the action in the presence of these collective coordinates gives terms for coupled triaxial rotors [12], with moments of inertia functionals of the time varying fields [20]. For the lowest-lying states the rotors decouple [12,13] and simplify. The rotor correction to the quantization condition [21,22] for the $s=1$, $i=0$ state is

$$W(E_{\rm cl}) - \frac{e^4}{2}\int_0^{T(E_{\rm cl})} dt \left\{ \frac{1}{V_{11}(t)} - \frac{1}{V_{11}(T(E_{\rm cl}))} \right\} 1(1+1) = \pi e^2 ,$$

where $E_{\rm cl} = -dS/dT$, and where we follow the notation of Ref. [13] for the moments of inertia. From the $E_{\rm cl}$ that satisfies the quantization condition, the quantum energy is

$$E = E_{\rm cl} + e^4 \frac{1(1+1)}{2V_{11}(T(E_{\rm cl}))} .$$

Similarly, for the $s=0, i=1$ state, the result is similar with $V_{11} \to U_{11}$. For a given value of e^2, these corrections to W push the bound-state energy higher. Because $V_{11} > U_{11}$, the $s=1, i=0$ state is lower in energy than the $s=0, i=1$ state. Because V_{11} and U_{11} at $T(E_{\rm cl})$, corresponding to the largest separation, should scale as the square of the effective radius, the splitting of these states should be $\sim \frac{1}{4}$ that of the static solution.

For large e^2 our semiclassical quantization yields a picture of the $B=2$ system which is a weakly bound pair of nucleons, well separated for most of their period, with small energy splittings and a large quadrupole moment. The consistency of these qualitative features with experiment gives additional support to the conjecture that $1/N$ is a good expansion parameter for QCD.

We thank L. Carson, S. Libby, J. Hughes, M. Prasad, and J. Wambach for valuable discussions. This work was performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

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[20] In the expansion of the action about the dynamical solution, after introducing collective coordinates for global rotations and isorotations, the terms linear in the time derivative of the collective coordinates (and linear in the time derivative of the classical solution) vanish due to total angular momentum and isospin conservation.

[21] Within the DHN quantization framework [9], it is natural to quantize in path-integral language. We generalize the path-integral rotor quantization of R. Schulman, Phys. Rev. **176**, 1558 (1968), to the case of a time-dependent moment of inertia.

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