New Type of Intermittency in Discontinuous Maps

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(Received 12 September 1991)

Intermittent behavior originating in a point of discontinuity in 1D maps is investigated. Studying the duration of the laminar phase, we find a logarithmic dependence of the average laminar length $\langle l \rangle$ on the control parameter ϵ in contrast to the three conventional types of intermittency characterized by power-law scaling. Analytical considerations give the relation $\langle l \rangle = \log(\epsilon)/\log(s) + \beta$ (where s is the "slope" at the point of discontinuity). Numerical data obtained from a relaxation oscillator model are in good agreement with these results.

PACS numbers: 05.45.+b

The intermittency route to chaos is characterized by the intermittent change between long, almost regular (so-called laminar) periods and shorter chaotic bursts. The scaling properties of the laminar lengths were studied for the first time by Pomeau and Manneville [1,2] in the Lorenz model. The intermittent behavior originates in the influence of a fixed point x_f that becomes unstable at a critical control-parameter value r_c. For higher values of the control parameter intermittent behavior is observed. The laminar behavior takes place in the vicinity of x_f in phase space. After the system has escaped from the influence of x_f, chaotic behavior occurs until the system again reaches the laminar region (is reinjected).

When studying intermittency the properties of the laminar period are normally treated independently of the mechanism that generates the chaotic phases. The average length of the laminar phase $\langle l \rangle$ decreases as the control parameter r increases. One finds the scaling relation $\langle l \rangle \propto \epsilon^{-\alpha}$, where ϵ is the distance $\epsilon = r - r_c$ from the critical control parameter. Pomeau and Manneville distinguish between three different types of intermittency, depending on the way the fixed point becomes unstable, i.e., in the way the eigenvalues of the map cross the unit circle at r_c [3]. This classification uses the fact that the corresponding Poincaré mapping is differentiable around the fixed point.

Properties of mappings that are not differentiable around the fixed point, however, are not covered by this classification, even though it is not only "exotic" systems [4-11] that show discontinuous mappings. In this kind of map a fixed point can lose its stability in a fourth way, that is, by colliding with a point of discontinuity. At the critical control parameter $\epsilon = 0$ this point is stable in one direction of the 1D phase space and unstable in the other; for higher values of ϵ laminar behavior can be observed in the vicinity of the formerly stable part.

In this Letter we study the laminar lengths of intermittent time series in such discontinuous maps. In order to do this, we discuss as an example the laminar lengths generated by quadratic maps with one point of discontinuity and a stochastic reinjection [12] into the laminar region. The results are compared with numerical data obtained from a model of an electronic relaxation oscillator [9].

For the numerical simulations we construct a mapping $x_{n+1} = f(x_n)$ that changes from a quadratic iteration $f_1(x)$ to a random reinjection $f_2(x)$ at a point of discontinuity x_d (see Fig. 1); above the point x_d the map is differentiable, and for $x < x_d$ the iterations are reinjected randomly into the smooth region. Such a map may read

$$f(x) = \begin{cases} f_1(x) = (x-a)^2 + b & \text{for } x \geq x_d\,, \\ f_2(x) = A\,\mathrm{rnd}(x) + B & \text{for } x < x_d \end{cases} \tag{1}$$

(modulo 1), where rnd(x) is a random number from the interval [0,1], and A and B are chosen such that the reinjection takes place below the intersection of the function $f_1(x)$ with the diagonal $x_{n+1} = x_n$. The shift parameters a and b arrange the dependence of the parabola $f_1(x)$ on the control parameter ϵ; with decreasing ϵ the value $f_1(x_d)$ approaches the diagonal $x_{n+1} = x_n$ [giving a fixed point $f(x_d) = x_d$ for $\epsilon = 0$] and thus we set $\epsilon = f(x_d) - x_d$. In principle, this could be achieved if a is kept con-

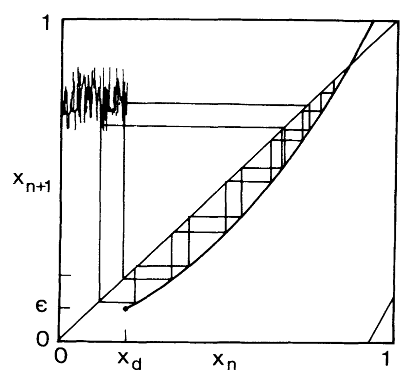

FIG. 1. The considered mapping $x_{n+1} = f(x_n)$ with a point of discontinuity at $x_d = 2\epsilon$ and the parameters $\epsilon = 0.1$, $s = 0.5$. Additionally, the path of one iteration is shown; the iteration is reinjected stochastically into the laminar region after passing x_d.

stant and only the vertical shift b is varied according to ϵ. For our numerical simulations we use a more suitable choice of a and b according to the following conditions: The "slope" $df_1(x_d)/dx$ at the point of discontinuity x_d is adjusted independently of ϵ, i.e., $df_1(x_d)/dx = s$. Additionally, for numerical reasons, it will be appropriate to choose $f(x_d)$ of the same order of magnitude as ϵ. Choosing $f(x_d) = \epsilon$ determines the position of the discontinuous point as $x_d = 2\epsilon$. The two conditions can be fulfilled by setting the dependence of a and b on the control parameter ϵ and on the slope s to

$$a = (4\epsilon - s)/2, \quad b = \epsilon - (2\epsilon - a)^2 = \epsilon - s^2/4 . \qquad (2)$$

We want to emphasize that this specific form of the mapping is only one example of a large class of mappings with similar properties. Choosing this form has the advantage that the range accessible to numerical simulation is very large due to the fact that for small values of ϵ no additional offset has to be handled during the iteration. Thereby it becomes possible to use as the whole range of magnitude for the control parameter that which is covered by the representation of numbers in the computer [13]. It should also be mentioned that the reinjection can take place anywhere in the basin of attraction of x_d, even though the duration of the laminar period for small values of ϵ depends almost entirely on the map in the vicinity of x_d.

We iterate this map for different values of the control parameter ϵ averaging the number of iterations l in the laminar region over 500 reinjections. The result shows clearly that there occurs no power-law dependence of the average laminar length on the control parameter as would have been expected for the conventional types of intermittency. As shown in Fig. 2(a) we observe a logarithmic dependence of $\langle l \rangle = \alpha \log(\epsilon) + \beta$ for nonzero values of the slope s, where the factor α decreases with the slope s.

This logarithmic dependence can be understood if we approximate the behavior of the simulated quadratic mapping by a linear map of the form $f(x) = sx$, and calculate the number of iterations that it takes to pass the channel and to reach $x_d = 2\epsilon$. Starting the iteration with some initial value x_0 we find the nth iterate to be $x_n = s^n x$ and therefore we can calculate the number of iterations in the laminar region from the integer part of

$$l = \frac{\log(2\epsilon) - \log(x_0)}{\log(s)} . \qquad (3)$$

Assuming that the reinjection takes place randomly in a constant interval around x_0, in the limit of $\epsilon \to 0$ we can directly use the real number l as an approximation of the behavior of the average laminar length $\langle l \rangle$,

$$\langle l \rangle = \frac{1}{\log(s)} \log(\epsilon) + \beta(s) , \qquad (4)$$

with some offset value β. Thus the lines of Fig. 2(a) should have the slope $1/\log(s)$. We have compared this dependence on s with the numerical data obtained from

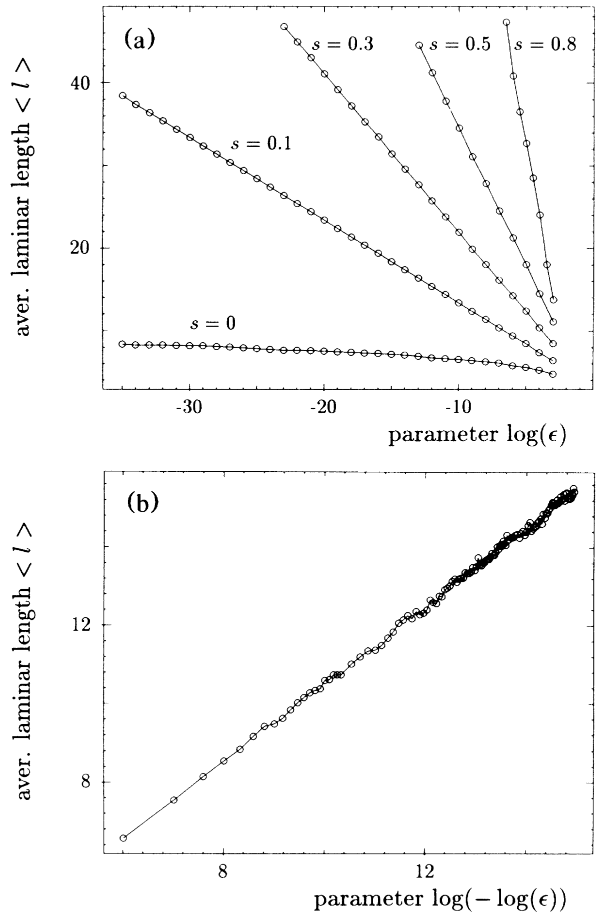

FIG. 2. The average laminar length for different slopes s of the map. (a) $\langle l \rangle$ is plotted vs $\log_{10}(\epsilon)$. The linear shape of the resulting plot for nonzero slopes s indicates that the relation (4) is true. (b) $\langle l \rangle$ is plotted vs $\log_{10}[-\log_{10}(\epsilon)]$ for $s = 0$ verifying the relation (6).

the original quadratic map (1) for several slopes s in the range $]0,1[$ and found that the results are in good agreement with (4).

It has to be mentioned that this analytical result holds only if the slope s is constant and the reinjection takes place around a constant value x_0. If these values are a function of the control parameter ϵ the dependence $s(\epsilon)$ and $x_0(\epsilon)$ would be superimposed on the relation (3). However, normally this superimposed function will not lead to a power-law dependence of $\langle l \rangle$ on ϵ.

The relation (4) is undefined for $s = 1$ and 0. In the case of $s = 1$ we can use a similar linear treatment to find $\langle l \rangle = \alpha \epsilon^{-1} + \beta$. For $s = 0$ we have to use a nonlinear ansatz. We can consider for instance the map $f(x) = x^2 - \epsilon$ and ask how many iterations n are necessary such that $x_n \le 0$. This is equivalent to the question of when $f(x) = x^2$ reaches ϵ and thus yields

$$l = \frac{1}{\log(2)} \log\left(\frac{\log(\epsilon)}{\log(x_0)}\right) , \qquad (5)$$

or again more generally

$$\langle l \rangle = \alpha \log[-\log(\epsilon)] + \beta \,. \qquad (6)$$

This relation is checked in Fig. 2(b) where $\langle l \rangle$ is plotted versus $\log[-\log(\epsilon)]$ for the numerical data obtained from our original map (1) showing very good agreement with (6).

In principle a piecewise linear treatment that results in (4) is applicable for all kinds of maps that have a nonzero first-order Taylor expansion [14] around the fixed point, as, e.g., the considered map for $s \neq 0$. For $s=0$ (where the first-order expansion vanishes) we had to use a quadratic ansatz that results in (6) where the factor $1/\log(2)$ in (5) represents the second-order approximation.

In order to check that the described type of intermittency cannot only be found in constructed mappings but also in physical systems, we compare our results to the results obtained from the numerical simulation of a relaxation oscillator. This oscillator may be realized by means of a thyratron circuit [9]. The corresponding map turns out to be a concatenation of a circle map and an inverse circle map that can be constructed from the underlying differential equation [11,15]. For the considered parameters a 12-cycle becomes unstable due to intermittency. In the inset of Fig. 3 an enlargement of the corresponding multimap $f^{12}(x)$ is plotted. We find a situation that is quite similar to that of Fig. 1, where a fixed point has vanished due to its collision with a point of discontinuity. We iterate this mapping for different values of the control parameter (driving voltage U) and plot (Fig. 3) the resulting laminar lengths versus the logarithm of the distance $\epsilon = U - U_0$, where U_0 is the critical control parameter. Again we observe that the relation (4) is reproduced very well.

Even though a discontinuous map can be approximated by a continuous map with a high derivative at x_d, the described behavior is in fact different from type-I intermittency. The continuous approximation will show the power-law behavior of $\langle l \rangle$ in the limit $\epsilon \to 0$. However, for larger length scales of ϵ—beginning with an ϵ value that depends on the degree of approximation—one finds again the reported logarithmic behavior. In the limit of the nondifferentiable mapping the type-I power-law behavior vanishes and only the logarithmic behavior remains.

We would like to mention that in some cases the distinction between the case of power-law scaling and the logarithmic dependence of $\langle l \rangle$ on ϵ might take special care. On the background of this investigation one cannot assume *a priori* a power-law behavior of the laminar lengths if the properties of the underlying Poincaré maps are unknown. This might be especially important in the analysis of experimental data, where often only a few orders of magnitude of the control parameter can be analyzed and the exact value of the critical control parameter r_c is not known. A small error Δr in the estimation of r_c

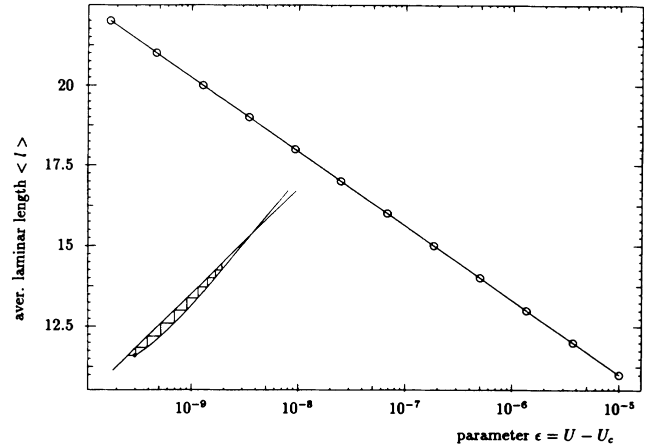

FIG. 3. Results for the thyratron relaxation oscillator. Average laminar length $\langle l \rangle$ for different values of the control parameter. Again the result of (4) can be confirmed. Inset: A part of the multimap $f^{12}(x)$ vs x, showing a situation quite similar to the constructed mapping in Fig. 1.

could result in an apparently linear dependence in the plot of $\log\langle l \rangle$ vs $\log(r - r_c + \Delta r)$. This result may be interpreted erroneously as an indication for one of the conventional types of intermittency.

The new type of intermittency presented in this Letter has to be considered in all systems that show discontinuous Poincaré mappings. In fact there is a large class of systems that exhibit such mappings. Relaxation oscillators [4,11] that can be described by discontinuous circle maps [5,6,9] and which are of practical interest in many different fields of science, e.g., in electronics [9,11,16], chemistry [17], biology [8,18], and neural networks [10], are typical examples.

Helpful discussions with P. Alstrom and B. Christiansen are gratefully acknowledged. This work is supported by the Deutsche Forschungsgemeinschaft via the Sonderforschungsbereich No. 185 "Nichtlineare Dynamik."

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[12] In principle the same results can be obtained with deterministic chaotic reinjection but random reinjection can be simulated and controlled much easier.

[13] For example, for $s=0$, ϵ can be 10^{-5000} with REAL*16 resolution.

[14] The expansion is considered for only the differentiable part of the map [in our example the part $f_1(x)$].

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