Complete Mode Locking in Models of Charge-Density Waves

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Mode locking in ac-driven charge-density waves is studied numerically in two and three dimensions, using both continuous equations of motion and a cellular automaton model. As the system size is increased, a complete devil's staircase of steps is approached. The fractal dimension of the set of gaps in the staircase is found to be $D_0=0.75\pm0.04$ (0.88 ± 0.05) in two (three) dimensions in the automaton model. The spectrum of singularities $f(\alpha)$ is calculated and is related to the dynamic critical exponent ζ for the charge-density-wave depinning transition.

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Extended systems with many degrees of freedom, such as charge-density waves (CDW's) and Josephson junction arrays, are often seen to exhibit mode-locking behavior experimentally [1]. Mode locking occurs when the system locks into a rational multiple of the external drive frequency. As a parameter is varied, the system goes through a sequence of such Shapiro steps. This behavior results from a collective phenomenon which is not yet well understood.

A useful framework for the study of mode locking has been developed and used in the context of nonlinear dynamical systems with few degrees of freedom, such as the circle map and the Frenkel-Kontorova model [2-4]. The winding number w as a function of some driving force Ω exhibits a devil's-staircase (DS) behavior: There is a mode-locked plateau for any rational $w=p/q$. On a critical line in parameter space, the gaps between the steps have measure zero and the DS is said to be complete. It was found numerically [2], and later confirmed experimentally [5], that the Hausdorff dimension of the set of gaps on this critical line, $D_0\simeq0.87$, is a universal number for a class of quasiperiodic maps. Generalized fractal dimensions and the spectrum of singularities $f(\alpha)$ for such sets have also been studied [3,4].

In this Letter, we report results from the numerical study of the mode-locked steps for sliding charge-density waves. CDW's are modeled as elastic media subject to an external drive force with dc and ac components and to spatially random pinning forces. Using both cellular automaton and more complete models, we find strong evidence that, for the pulsed drive fields we study, a complete devil's-staircase behavior of mode-locked steps is typical. On each step the velocity of the CDW is rationally related to the ac drive period and, as the size of the system is increased, the number of mode-locked steps increases, suggesting that all rational steps will be seen in an infinite system. We calculate the Hausdorff dimension D_0 of the set of gaps and obtain $D_0=0.75\pm0.04$ (0.88 ± 0.05) for the cellular automaton model in two (three) dimensions, respectively. In the continuous time model, we find a fractal dimension of $D_0=0.75\pm0.08$ in two dimensions, consistent with the automaton result. We calculate the singularity spectrum $f(\alpha)$ for the automaton and estimate $\alpha_{\max}-\alpha_{\min}=0.15\pm0.08$ ($d=2$), 0.09 ± 0.05 ($d=3$). The scaling properties of the CDW staircase are much more homogeneous than those of the circle map, for which $\alpha_{\max}-\alpha_{\min}\approx0.42$. We also find that the minimal scaling exponent, $\alpha_{\min}$, is equal within our errors to the dynamic critical exponent ζ for the velocity near the depinning transition. Previous simulations of the continuous CDW model in one dimension showed mode locking including high-order subharmonics [6,7], but, at least in one-dimensional automaton models [8], the DS behavior was not observed; we argue that in automaton models, a DS can only be seen in two or more dimensions.

A CDW is a condensate of charge carriers which has a periodic charge density that appears in a variety of highly anisotropic solids [9]. The CDW behaves as an elastic medium that is distorted by quenched random impurities in the solid [9,10]. At small applied electric fields, these impurities pin the CDW in a static configuration. At dc fields F exceeding a threshold field F_T, the CDW slides and carries a current proportional to its velocity v. The depinning transition separating these two behaviors has been studied as a dynamic critical phenomenon [11], with the velocity behaving as $v\sim(F-F_T)^\zeta$, for a critical exponent ζ.

We study two models for the dynamics of CDW's, which incorporate these characteristics of physical CDW's. The first model, which has been studied extensively numerically [9], is a simplification of the model due to Sneddon, Cross, and Fisher (SCF) [10]. The distortions of the CDW in this model are given by the variables

φ_i, which are defined at N lattice sites indexed by i. The equations of motion for the φ_i are found from the Fukuyama-Lee-Rice Hamiltonian, assuming relaxational dynamics, and in dimensionless units are given by

$$\dot{\varphi}_i = \Delta^2\varphi_i + h\cos(\varphi_i - \beta_i) + F(t)\,, \qquad (1)$$

where the term with the lattice Laplacian, Δ^2, defines the elastic forces, h is a pinning strength magnitude, the $\beta_i \in [0,2\pi)$ are random variables representing the quenched disorder, and $F(t)$ is the external force due to the electric field. This model reproduces the CDW depinning transition and the complex memory effects seen in CDW's [9]. We use periodic boundary conditions. The average velocity v is defined as $v = N^{-1}\sum\dot{\varphi}_i$ and is independent of initial conditions [12]. For small amplitude sinusoidal driving at frequency ω, the properties of the SCF model and Eq. (1) are subtly different. A narrow region of mode locking was found for Eq. (1) even for small ac fields [7], because for $\omega \to v$, the perturbation theory is stable in the SCF model, but not for Eq. (1) [6]. There are no significant differences between the models for pulsed driving, which we use here.

Coppersmith [8] has shown that, in the limit of strong pinning h and for pulsed drive fields $F(t) = F_0 + F_p\sum_n \delta(t - nt_0)$ (where F_p is the magnitude of the pulses, with the period t_0 longer than local relaxation times, and $n = 1,2,\ldots$), the equations of motion, Eq. (1), are well approximated by an automaton model. In the automaton model, the phase takes on discrete values (representing the positions of the minima of the pinning potential) and the continuous time dynamics is replaced by a map between the configurations after successive pulses. Numerical work on the dynamics of Eq. (1) also suggests that the critical behavior at the depinning transition is independent of the details of the pinning [13]. With these points in mind, we have also studied a variant of the cellular automaton model in [8] which is given by the discrete-time map

$$m_i \to m_i + 1 \text{ when } \Delta^2 m_i + (2\pi)^{-1}\Delta^2\beta_i + \epsilon\Delta^2(\Delta^2 m_i) + (F-h) > 0\,, \qquad (2)$$

where the m_i are integers, $(F-h)$ is the total of the drive field ($F = F_0 + F_p$) minus the maximum pinning force that must be overcome, and the $\Delta^2\beta_i$ term is due to the disorder. The term with the coefficient ϵ gives rise to next-nearest-neighbor interactions and represents higher-order terms in a F_0/h expansion. The parameter ϵ can be used to study the effects of finite pinning strength and also allows us to investigate the sensitivity of our results to variations in the model. The average velocity v is given by the average fraction of sites where m_i advances in a single time step.

We note that in both the continuous model with a long pulse period and the automaton model, if there is no quenched disorder ($\beta_i \equiv 0$ or β_i periodic in space) there are only a finite number of mode-locked steps. The case where β_i is quasiperiodic is currently being studied.

A Thinking Machines CM-2 with 16K processors was used to simulate the equations of motion, Eqs. (1) and (2). Mode-locked behavior was simply defined as the configuration returning to the same shape after an integral number of drive periods (to within numerical error for the continuous model). Bisection in the dc field was used to find the endpoints of mode-locked steps to a high accuracy (for the automaton, to one part in 10^7). Steps were found for all rationals in successive generations of the Farey construction [14]. The gth generation $\mathcal{F}_g$ in the Farey tree is given by the set of 2^g+1 rationals $\{p_n^g/q_n^g\}$ ordered by magnitude, with initial set $\mathcal{F}_0 = \{0/1, 1/2\}$, and $\mathcal{F}_{g+1}$ being the union of $\mathcal{F}_g$ and the set $\{(p_n^g + p_{n+1}^g)/(q_n^g + q_{n+1}^g)$, for $n = 1,\ldots,2^g\}$. (Due to a symmetry in the automaton model, we consider only the rationals in the interval $[0/1, 1/2]$.)

The mode-locking behavior for the cellular automaton model in dimension $d=2$ is shown in Fig. 1, where we have plotted the average velocity as a function of the parameter $(F-h)$, for a system of size 256^2 and $\epsilon = 0.1$. The steps shown are for all rationals down to generation $g=7$ of the Farey tree, i.e., 129 steps. The whole interval in $(F-h)$ is covered by mode-locked steps (not shown here) [15]. As the size of the system is increased, the number of mode-locked steps increases and the Farey tree becomes complete to more generations, with approximately one more complete generation seen for each doubling of the linear size of the system. This suggests that

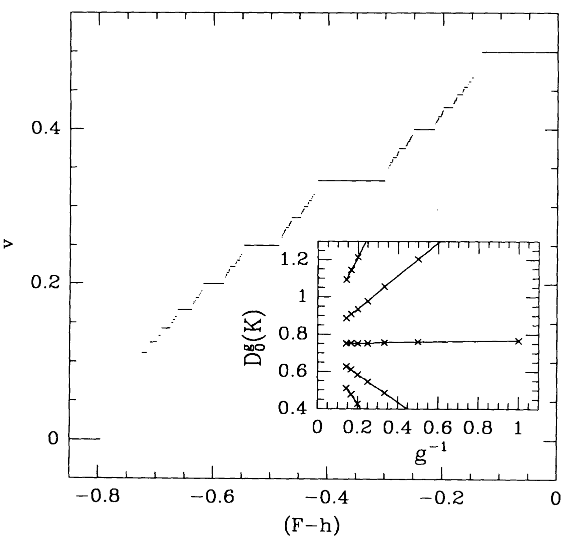

FIG. 1. Mode-locking steps for 256^2 automaton with $\epsilon = 0.1$, shown as average velocity v vs the barrier height $(F-h)$. All steps are shown down to the $g=7$ level of the Farey tree of rationals. The inset shows, for various K, the estimates $D_q^g(K)$ vs g^{-1}, for $q=0$ in the two-dimensional automaton model. Extrapolation to $g^{-1}=0$ gives the Hausdorff dimension $D_0 = 0.75 \pm 0.04$.

in the infinite system, all rational velocities will be seen, and that the mode-locking forms a DS.

To analyze the DS we calculate the Hausdorff dimension D_0 and the generalized dimensions D_q of the set of gaps [3]. The measure P_i associated with the gap between mode-locked steps p_i/q_i and p_{i+1}/q_{i+1} is defined as the difference in the velocities across the gap, i.e., $P_i = p_i/q_i - p_{i+1}/q_{i+1}$. The width of the gap l_i is the difference in applied dc field F at the right end of the step with velocity p_i/q_i and the left end of the step with velocity p_{i+1}/q_{i+1}. For any constant K, the approximation $D_q^g(K)$ to D_q associated with the gth generation of the Farey tree is given implicitly by [3]

$$K = \sum_{i=1}^{2^g} \frac{P_i^q}{l_i^{(q-1)D_q^g}} . \tag{3}$$

For any finite $K > 0$, the generalized dimension D_q is defined as $\lim_{g\to\infty} D_q^g(K) = D_q$. We find D_q by plotting $D_q^g(K)$ vs g^{-1} for various K and extrapolating to the limit $g^{-1} \to 0$.

In the inset in Fig. 1, we show such plots for a two-dimensional automaton of size 256^2, with $\epsilon = 0.1$. By extrapolating to zero, we estimate D_0 to be 0.75 ± 0.04, in agreement with our result for $\epsilon = 0$, which is $D_0 = 0.73 \pm 0.04$. From a similar analysis for the $d = 3$ automaton, we determine $D_0 = 0.88 \pm 0.05$.

We have also examined mode-locked steps in the continuous model Eq. (1) in two dimensions. A plot of v vs F_0 for pulsed fields, with $F_p = 1$, $t_0 = 4$, and a pinning strength of $h = 25$, is shown in Fig. 2. Again, we find complete mode locking to a relatively large number of generations in the Farey tree ($g = 6$) and many other steps with higher generation numbers. For all values of F_0 we have examined, we find mode locking. We obtain $D_0 = 0.75 \pm 0.08$ for this system, which agrees, within our errors, with the value for the two-dimensional automaton.

Using our results for D_q, we determine the spectrum of singularities $f(\alpha) = q(d/dq)[(q-1)D_q] - (q-1)D_q$, which describes the scaling properties of the fractal [3]. This function [16], for the $d = 2$ automaton, is shown in Fig. 3. The points where $f(\alpha) = 0$ have special significance. The largest value, $\alpha_{\max} = D_{-\infty}$, is determined by the gaps for which the measure P_i is very small. Gaps of this type appear in the vicinity of quadratic irrationals. On the other hand, the smallest value $\alpha_{\min} = D_\infty$, is determined by gaps for which the measure P_i is very large. Such gaps are those adjoining low-generation steps. The measure of these gaps, $P_i \sim 1/g$, decays more slowly than any other gaps. Therefore, high-order gaps in the vicinity of primary steps scale like $P_i \sim l_i^{\alpha_{\min}}$, where l_i is the width of the gap on the dc field axis, while P_i is its measure along the velocity axis. The relation between the velocity and the field near the 0/1 gap is also described by $v \sim (F - F_T)^\zeta$ where we find the critical exponent $\zeta = 0.64 \pm 0.03$, 0.81 ± 0.03 in $d = 2,3$, respectively (in numerical agreement with results on other CDW models [13]). Since both relations become exact in the scaling regime where very-high-order steps are very narrow and can be smoothed out we find that $\alpha_{\min} = \zeta$. Universality of the critical behavior suggests the same power law ζ describes the behavior near other steps (a related argument gives $\alpha_{\min} = 1/2$ for the circle map [2]); we have confirmed this near the 1/2 and 1/3 steps. Numerically

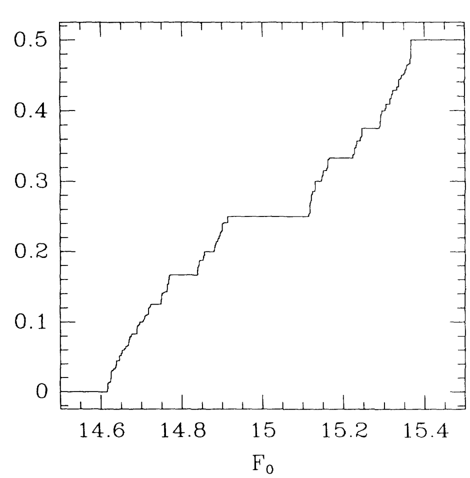

FIG. 2. Mode-locking steps shown for the continuous CDW model of Eq. (1) in a system of size 64^2 in a pulsed field (pinning strength $h = 25$ and a pulse magnitude of $F_p = 1$). The entire velocity v vs dc field F_0 is shown; in this finite system, there are steps for all rationals down to the $g = 6$ level of the Farey construction.

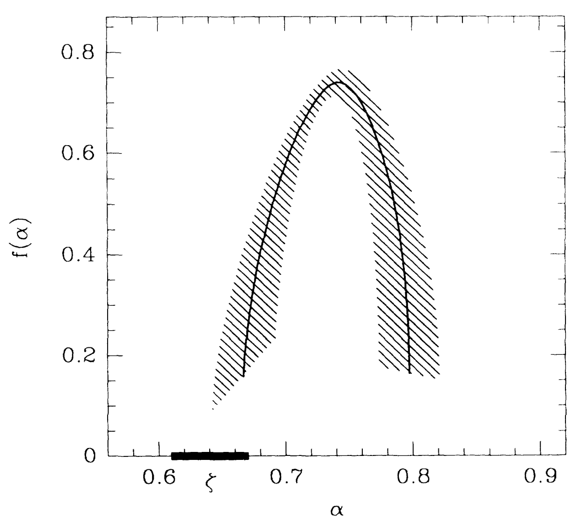

FIG. 3. The spectrum of singularities $f(\alpha)$ for the devil's staircase of Fig. 1. The shaded region shows the uncertainties in our analysis. The smallest scaling exponent $\alpha_{\min}$ is found to be equal, within our numerical uncertainties, to the dynamic critical exponent ζ, which is represented by the thick line on the α axis.

we find $\alpha_{\min} = 0.67 \pm 0.04$, 0.80 ± 0.05, in $d = 2,3$, respectively, consistent with the relation $\alpha_{\min} = \zeta$.

In the one-dimensional cellular automaton with $\epsilon = 0$, we find that the mode-locking behavior is nearly trivial, with only half-integer steps occurring. Coppersmith [8] found only a few mode-locked steps in a more complicated automaton model. The simple behavior in one-dimensional systems is related to the behavior of closely related sandpile models [17]. The depinning transition at the edge of the 0/1 step is analogous to the first avalanche in the sandpile model that is the size of the system. Because of the periodic boundary conditions, this avalanche does not die out and the average velocity is easily shown to be 1/2 immediately above the transition. The relation between 1D automata and the 1D continuum model (which does have high-order subharmonics [6]) is puzzling.

In summary, we have investigated mode locking in models of CDW's, which are extended driven systems with quenched disorder. We find that mode locking to high-order rationals is typical for certain ac drives in two and three dimensions, with the number of steps increasing with system size, and estimate the fractal dimensions of the gaps. Disorder is necessary to the existence of the high-order steps. We find that the spectrum of singularities $f(\alpha)$ is narrow compared to that for few-degrees-of-freedom systems. We have studied pulsed driving, with a period long in comparison to relaxation times and always find a complete DS. If the pulses are moved close together or replaced by sinusoidal driving, the DS is expected to become incomplete. Further simulations are currently being performed to study these cases.

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[1] S. P. Benz, M. S. Rzchowski, M. Tinkham, and C. J. Lobb, Phys. Rev. Lett. **64**, 693 (1990); J. Richard, P. Monceau, and M. Renard, Phys. Rev. B **25**, 948 (1982); S. E. Brown, G. Mozurkewich, and G. Grüner, Phys. Rev. Lett. **52**, 2277 (1984); R. E. Thorne, J. R. Tucker, and J. Bardeen, Phys. Rev. Lett. **58**, 828 (1987); S. Bhattacharya, J. P. Stokes, M. J. Higgins, and R. A. Klemm, Phys. Rev. Lett. **59**, 1849 (1987); J. Levy and M. S. Sherwin, Phys. Rev. Lett. **67**, 2846 (1991).

[2] M. H. Jensen, P. Bak, and T. Bohr, Phys. Rev. Lett. **50**, 1637 (1983); Phys. Rev. A **30**, 1969 (1984); **30**, 1970 (1984).

[3] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[4] O. Biham and D. Mukamel, Phys. Rev. A **39**, 5326 (1989); W. Wenzel, O. Biham, and C. Jayaprakash, Phys. Rev. A **43**, 6550 (1991).

[5] J. Stavans, F. Heslot, and A. Libchaber, Phys. Rev. Lett. **55**, 596 (1985).

[6] S. N. Coppersmith and P. B. Littlewood, Phys. Rev. Lett. **57**, 1927 (1986).

[7] H. Matsukawa and H. Takayama, Jpn. J. Appl. Phys. **26**, 601 (1987).

[8] S.N. Coppersmith, Phys. Rev. A **36**, 311 (1987); **38**, 375 (1988).

[9] *Charge Density Waves in Solids*, edited by G. Hutiray and J. Sólyom (Springer-Verlag, Berlin, 1985); G. Grüner, Rev. Mod. Phys. **60**, 1129 (1988); *Charge Density Waves in Solids*, edited by L. P. Gorkov and G. Grüner (Elsevier, New York, 1989).

[10] H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1978); P. A. Lee and T. M. Rice, Phys. Rev. B **19**, 3970 (1979); K. B. Efetov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **72**, 2350 (1977) [Sov. Phys. JETP **45**, 1236 (1977); L. Sneddon, M. C. Cross, and D. S. Fisher, Phys. Rev. Lett. **49**, 292 (1982).

[11] D. S. Fisher, Phys. Rev. Lett. **50**, 1486 (1983); Phys. Rev. B **31**, 1396 (1985).

[12] Though the velocity is unique, there are many distinct periodic states at a given field in *ac-driven* CDW's. See A. A. Middleton, Phys. Rev. Lett. **68**, 670 (1992).

[13] D. S. Fisher and A. A. Middleton (to be published); A. A. Middleton, Ph.D. thesis, Princeton University, 1990 (unpublished).

[14] See, e.g., I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers* (Wiley, New York, 1960).

[15] This covering is not unexpected, as there are only a finite number of states in a finite automaton, so that the dynamics must be periodic. In general, however, the period can be as large as an exponential of the system size [e.g., see K. Wiesenfeld, J. Theiler, and B. McNamara, Phys. Rev. Lett. **65**, 949 (1990)]. In contrast, we find the typical period to be relatively short and that the width of the steps converges as the system size increases.

[16] To determine $f(\alpha)$ without calculating numerical derivatives, we have generalized the method of A. Chhabra and R. V. Jensen [Phys. Rev. Lett. **62**, 1327 (1989)] to nonuniform intervals. After determining $\tau(q) = (q-1)D_q$, $f(\alpha)$ is found by the relations $f(q) = \langle \ln(P_i^q / l_i^\tau) \rangle / \langle \ln l_i \rangle$ and $\alpha(q) = \langle \ln P_i \rangle / \langle \ln l_i \rangle$, where the weighted average over gaps is given by $\langle x_i \rangle = \sum_i x_i P_i^q / l_i^\tau$.

[17] J. M. Carlson, J. T. Chayes, E. R. Grannan, and G. H. Swindle, Phys. Rev. A **42**, 2467 (1990).