Unlimited Particle Acceleration by Waves in a Magnetic Field

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A general description is given to the unlimited acceleration of charged particles by a wave in the presence of a uniform magnetic field. The wave can have an arbitrary mixture of electrostatic and electromagnetic components and the propagation direction in the magnetic field is also arbitrary. Exact limits on parameters for trapping and acceleration are derived. The probability of trapping in a gyration period is also presented, and it is shown that a particle once trapped remains so and undergoes unlimited acceleration.

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The problem of the acceleration of charged particles by the combination of a wave (electrostatic or electromagnetic) and a uniform magnetic field has recently attracted much attention [1-6]. In addition to its interest in basic physics, its applications to particle accelerators [2,7], plasma physics, and as a possible mechanism for cosmic ray generation in pulsar environments should be noted [8].

There are two broad classes of acceleration mechanisms. In the first, the particle is trapped by the wave, requiring a phase velocity $V=\omega_0/\kappa_0<c$; in the second, the acceleration is stochastic [6], with $V\geq c$. Here we concentrate on the first type, the coherent acceleration process. Many authors have considered the case when the wave propagates perpendicular to the magnetic field and is either electrostatic or electromagnetic. The general case where the wave propagates obliquely and has a mixture of electrostatic and electromagnetic components is the typical situation encountered when the wave propagates in a plasma environment. Karimabadi and coworkers [4] treated this case in a sequence of papers in the limit where the static magnetic field is strong and the wave field can be treated as a perturbation. In this limit the effect of trapping into resonance is absent. Here we show that the general case can be easily treated without such an approximation, and one finds well-defined limits in the parameters (angle of wave vector to magnetic field, wave amplitude, and magnetic field) where acceleration by wave trapping is possible. This kind of coherent acceleration has been recently observed in numerical calculations [4].

The wave is described by $\mathbf{E}=\mathbf{E}_0e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0 t)}$, where $\mathbf{E}_0$ has a parallel component $\mathbf{E}_0\cdot\mathbf{k}_0=E_\parallel k_0$, and the perpendicular component is typically elliptically polarized. The angle between $\mathbf{k}_0$ and the constant magnetic field $\mathbf{B}_0$ is α. We now perform a Lorentz transformation into the wave frame ($V<c$), where all the fields are static, the perpendicular components of the wave electric field vanish, while $E_\parallel$ is unchanged, the wave magnetic field and $\mathbf{B}_0$ are modified, and a static uniform electric field $\gamma_0\mathbf{V}\times\mathbf{B}_0$ emerges, where $\gamma_0=(1-V^2/c^2)^{-1/2}$. If the wave propagates in the x direction, and $\mathbf{B}_0$ is in the x-z plane, we have in the wave frame

$$\mathbf{B}=B_0\cos(\alpha)\hat{\mathbf{e}}_1+\frac{b_1}{\gamma_0}\sin(kx)\hat{\mathbf{e}}_2+\left[\gamma_0B_0\sin(\alpha)+\frac{b_2}{\gamma_0}\cos(kx)\right]\hat{\mathbf{e}}_3\,, \tag{1}$$

$$\mathbf{E}=E_\parallel\sin(kx)\hat{\mathbf{e}}_1-\gamma_0B_0V\sin(\alpha)\hat{\mathbf{e}}_2\,, \tag{2}$$

where b_1 and b_2 are the wave magnetic field amplitudes in the laboratory frame, and $k=(k_0^2-\omega_0^2/c^2)^{1/2}$. The equations of motion are

$$\frac{d}{dt}(\gamma v_x)=\epsilon\sin(kx)+v_y\left[\gamma_0\Omega_0\sin(\alpha)+\frac{\Omega_2\cos(kx)}{\gamma_0}\right]-v_z\frac{\Omega_1}{\gamma_0}\sin(kx)\,, \tag{3}$$

$$\frac{d}{dt}(\gamma v_y)=-\gamma_0\Omega_0\sin(\alpha)(V+v_x)-v_x\frac{\Omega_2}{\gamma_0}\cos(kx)+v_z\Omega_0\cos(\alpha)\,, \tag{4}$$

$$\frac{d}{dt}(\gamma v_z)=v_x\frac{\Omega_1}{\gamma_0}\sin(kx)-v_y\Omega_0\cos(\alpha)\,. \tag{5}$$

Here $\mathbf{v}$ is the particle velocity in the wave frame, $\gamma=(1-v^2/c^2)^{-1/2}$, $\epsilon=qE_\parallel/m$, $\Omega_0=qB_0/m$, $\Omega_{1,2}=qb_{1,2}/m$.

The energy conservation equation $mc^2d\gamma/dt=q\mathbf{E}\cdot\mathbf{v}$ yields

$$\frac{d}{dt}\left[c^2\gamma+\gamma_0\Omega_0\sin(\alpha)Vy+\frac{\epsilon}{k}\cos(kx)\right]=0\,, \tag{6}$$

while Eq. (5) gives

$$\frac{d}{dt}\left[\gamma v_z+\frac{\Omega_1}{\gamma_0 k}\cos(kx)+\Omega_0\cos(\alpha)y\right]=0\,. \tag{7}$$

One eliminates y from Eqs. (6) and (7) to find

$$\frac{d}{dt}\left[\gamma\left(\frac{c^2}{\gamma_0 V\tan(\alpha)}-v_z\right)+\left(\frac{\epsilon}{V\tan(\alpha)}-\Omega_1\right)\frac{\cos(kx)}{k\gamma_0}\right]=0. \quad (8)$$

We are looking now for unlimited acceleration where γ grows and asymptotically $\gamma\to\infty$, so

$$v_z\to v_{z0}=c^2/\gamma_0 V\tan(\alpha). \quad (9)$$

Since $v_{z0}<c$, we get our first condition for unlimited acceleration [4]

$$\tan(\alpha)\ge c/\gamma_0 V. \quad (10)$$

Introduce now the dimensionless parameter $\lambda=v_{z0}/c=c/\gamma_0 V\tan(\alpha)$ and write (4) in the asymptotic case ($v_z=v_{z0}<c$, $v_x\ll V$),

$$\frac{d}{dt}\left(\gamma v_y+\frac{\Omega_2}{\gamma_0 k}\sin(kx)\right)=-\gamma_0\Omega_0 V\sin(\alpha)(1-\lambda^2). \quad (11)$$

One expects v_y to approach a nonzero limit (the asymptotic case $v_y\to 0$ will be considered in another paper) since the acceleration is caused by the uniform electric field accelerating the particle in the y direction. Consequently, as $\gamma\to\infty$, $\gamma v_y\gg\Omega_0/\gamma_0 k$ and Eq. (11) can be integrated to yield

$$\gamma v_y=-\gamma_0\Omega_0 V\sin(\alpha)(1-\lambda^2)t+\text{const}, \quad (12)$$

where again in the asymptotic limit the constant can be ignored. In the same limit Eq. (7) gives ($v_z=v_{z0}$)

$$v_y=-\frac{c^2}{\Omega_0\gamma_0 V\sin(\alpha)}\frac{d\gamma}{dt}. \quad (13)$$

Substituting this into (12) gives

$$\gamma=[\gamma_0\Omega_0\sin(\alpha)V/c](1-\lambda^2)^{1/2}t, \quad (14)$$

leading to unlimited acceleration and

$$v_y\to v_{y0}=-c(1-\lambda^2)^{1/2}. \quad (15)$$

Now one substitutes v_{y0} and v_{z0} into Eq. (3) to find

$$\frac{d}{dt}(\gamma v_x)=\epsilon\sin(kx)-c(1-\lambda^2)^{1/2}\left(\gamma_0\Omega_0\sin(\alpha)+\frac{\Omega_2}{\gamma_0}\cos(kx)\right)-\frac{c\Omega_1\lambda}{\gamma_0}\sin(kx), \quad (16)$$

which one rewrites as

$$\frac{d}{dt}(\gamma v_x)+A\sin(kx+\varphi)+c(1-\lambda^2)^{1/2}\gamma_0\Omega_0\sin(\alpha)=0, \quad (17)$$

where

$$A=(1/\gamma_0)[(c\Omega_1\lambda-\gamma_0\epsilon)^2+c^2\Omega_2^2(1-\lambda^2)]^{1/2} \quad (18)$$

and

$$\varphi=\sin^{-1}[c\Omega_2(1-\lambda^2)^{1/2}/A\gamma_0]. \quad (19)$$

Equation (17) has the same structure as the one studied previously by Neishtadt and co-workers [5] and describes a particle with growing effective mass (γ) sliding on an incline with a periodic potential. The particle is trapped by the wave if v_x executes small oscillations, which requires

$$A>c(1-\lambda^2)^{1/2}\gamma_0\Omega_0\sin(\alpha). \quad (20)$$

This is the second condition on parameters for unlimited acceleration with trapping by the wave. Let us look now at specific examples.

(1) The wave is purely electrostatic, $\Omega_1=\Omega_2=0$. This leads to

$$\epsilon>c(1-\lambda^2)^{1/2}\Omega_0\gamma_0\sin(\alpha) \quad (21)$$

together with Eq. (10).

(2) The wave is purely transverse, linearly polarized with wave magnetic field in the z direction. Now $\epsilon=0$, $\Omega_1=0$, and the condition is

$$\Omega_2>\gamma_0^2\Omega_0\sin(\alpha). \quad (22)$$

From Eq. (1) this is also the condition for B_z having zeros in the wave frame. The physical meaning is interesting; the particle is trapped around a neutral layer of $B_z=0$.

(3) Transverse linearly polarized wave, with magnetic field in the y direction; $\epsilon=0$, $\Omega_2=0$, with the condition

$$\Omega_1>\gamma_0^2\Omega_0\sin(\alpha)[\gamma_0^2V^2\tan^2(\alpha)/c^2-1]^{1/2}. \quad (23)$$

While the first two modes accelerate particles when $\mathbf{k}\perp\mathbf{B}_0$ [in fact, with $\sin(\alpha)=1$ one recovers the results formerly available in the literature], the third mode excludes trapping if $\alpha=\pi/2$, $\tan(\alpha)\to\infty$.

It still remains to demonstrate that if the inequalities (10) and (20) are satisfied an initially untrapped particle will get trapped with a high probability. The Hamiltonian describing the asymptotic motion of Eq. (17) is

$$H=p^2/2\gamma(t)+\phi(x),$$
$$\phi(x)=-(A/k)[\cos(kx+\varphi)-\cos(kx_a+\varphi)]+c(1-\lambda^2)^{1/2}\gamma_0\Omega_0\sin(\alpha)(x-x_a), \quad (24)$$

where x_a is chosen in such a way that the potential is zero on the separatrix, and x_a is the saddle point as shown in Fig. 1. As time increases γ grows and so does the area enclosed by the separatrix as $p_{\text{sep}}\sim\sqrt{\gamma}$. At this point we assume the motion is adiabatic in the x-p plane. The

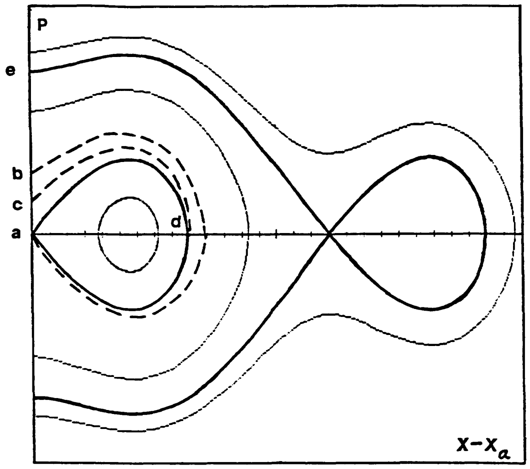

FIG. 1. Phase space of the Hamiltonian equation (24). Solid lines correspond to time and γ frozen; dashed lines, to γ evolving in time.

condition for adiabatic motion is [5]

$$(Ak)^{1/2} \gg \gamma_0 \Omega_0 V(1-\lambda^2)^{1/2} \sin(\alpha)/c\gamma^{1/2}.$$

This condition clearly holds at large values of γ; so a particle trapped at any time remains permanently trapped and undergoes unlimited acceleration.

Since the general equations describing the motion of the particle [Eqs. (3)–(5)] are nonintegrable, an untrapped particle can be expected to move chaotically in phase space until trapped (see Refs. [4,6]). Computational solutions of the exact equations confirm this assumption.

We calculate now the probability that a particle gets trapped in a given time. As γ slowly increases in time, a particle starting at some point b will spiral into point a as shown in the figure. So all particles starting on line segment a-b will spiral into the separatrix and get trapped. A particle midway between a and b, at point c, spirals into d.

The trapping probability is calculated by taking the ratio of particle fluxes that pass between a and b to those that pass between a and e. The flux of phase space area crossing a line spanned by points 1 and 2 is [9]

$$F_{12} = \int_1^2 |dl \times \dot{\mathbf{q}}| = H_2 - H_1 , \tag{25}$$

where dl is a line element on the phase plane and $\dot{\mathbf{q}} = (\dot{p}, \dot{x}) = (-\partial H/\partial x, \partial H/\partial p)$ is the flow velocity. The trapping probability is then

$$P = H_b/H_e . \tag{26}$$

Since from Eq. (24) $\dot{H} = -p^2\dot{\gamma}/2\gamma^2$,

$$-H_c = \int \dot{H}\, dt = \int_{x_a}^{x_d} \dot{H} \frac{dx}{v_x} = -\frac{\dot{\gamma}}{2\gamma} \int_{x_a}^{x_d} p\, dx . \tag{27}$$

The area inside the separatrix is $S = 2\int_{x_a}^{x_d} p_s\, dx = 2\int_{x_a}^{x_d} \sqrt{-2\gamma\phi}\, dx$, where p_s is the momentum on the separatrix and

$$\dot{S} = \int_{x_a}^{x_d} -\frac{2\phi\dot{\gamma}}{\sqrt{-2\phi\gamma}} dx = \frac{\dot{\gamma}}{\gamma} \int_{x_a}^{x_d} p_s\, dx . \tag{28}$$

So $H_b = 2H_c \cong \dot{S}$, and from Eq. (24)

$$H_e = H\left(x_a + \frac{2\pi}{k} \right) - H(x_a) = c(1-\lambda^2)^{1/2} \gamma_0 \Omega_0 \sin(\alpha) \frac{2\pi}{k} . \tag{29}$$

This leads to the general equation for the trapping probability,

$$P = \dot{S}k/2\pi c(1-\lambda^2)^{1/2} \gamma_0 \Omega_0 \sin(\alpha) . \tag{30}$$

Since S grows like $\sqrt{\gamma}$, and γ grows linearly with time as given in Eq. (14), it follows that $\dot{S} = S\gamma_0\Omega_0 \sin(\alpha) V/2c\gamma$, and therefore

$$P = \frac{Sk}{4\pi c\gamma} \frac{V}{c} = \frac{(\oint \sqrt{-2\phi}\, dx)k}{4\pi c\sqrt{\gamma}} \frac{V}{c} . \tag{31}$$

This equation has general validity, i.e., if the wave is not sinusoidal but deformed due to nonlinear effects, it is still valid. A relativistic particle ($\gamma \gg 1$) far from resonance with the wave moves along a Larmor orbit perturbed by a high-frequency oscillation caused by the wave and enters the resonance zone. Since $\dot{S} > 0$, capture is assured over many gyration periods, with the time $T = (P\Omega_0)^{-1}$.

In the laboratory frame, since S is a Lorentz invariant, $k = \gamma_0 k_0$, and $\gamma_{\rm lab} = \gamma_0 \gamma_{\rm wave}$, $P = Sk_0V/4\pi c^2 \gamma_{\rm lab}^{1/2} = S\omega_0/4\pi c^2 \gamma_{\rm lab}^{1/2}$. It is interesting to note that a slow particle ($\gamma \simeq 1$) has a higher probability to get trapped than a fast one with $\gamma \gg 1$.

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