Naked Singularities: Gravitationally Collapsing Configurations of Dust or Radiation in Spherical Symmetry, a Unified Treatment

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(Received 10 October 1991)

It is shown that the naked singular solutions that form in the gravitational spherical collapse of radiation are of the same nature as the ones which form in the collapse of dust matter. The stability, the strength, and other features of the singularities and Cauchy horizons are analyzed. This has implications in a possible formulation of the cosmic censorship hypothesis.

PACS numbers: 04.20.Jb

In the usual picture for the gravitational collapse of a spherical star, an event horizon appears first, followed by the formation of a spacetime singularity. In broad terms, the cosmic censorship hypothesis states that spacetime singularities are enclosed within an event horizon and cannot be seen by external observers, thus forbidding the existence of naked singularities [1]. If cosmic censorship fails to be valid, then unknown data may emerge from the singularity. In this case one cannot predict the future of the singularity and some form of determinism is lost even within classical general relativity. A precise mathematical formulation of cosmic censorship is still lacking, and there is a growing number of counterexamples to the above-mentioned version.

Naked singularities can form in the collapse of dust, soft matter, and null radiation [2]. These models are all spherically symmetric and such a high degree of symmetry could be used to invalidate on physical grounds the formation of those singularities. However, more recently, naked singularities have also appeared in the collapse of prolate configurations of collisionless particles [3]. This in turn shows that the naked singular behavior is not an artifact of special symmetries. Moreover, one can continue to consider spherical symmetry as representing a fair picture of what might be the actual physical process of a certain class of gravitational collapse. This is in the same spirit as what happened after the singularity theorem of Penrose [4] where one could neglect small deviations and consider spherical symmetry as a reasonable assumption.

Such naked singular solutions can be of two types, shell crossing and shell focusing, the latter being the more fundamental since it involves a curvature singularity where some curvature scalars diverge at some stage in the center of the collapsing matter. These shell-focusing singularities are not counterexamples to the hoop conjecture of Thorne or the event-horizon conjecture of Israel, as a spherical event horizon will always form afterwards, hiding the central singularity [5].

There are only two exact solutions of Einstein field equations which have been used to generate solutions with shell-focusing singularities. They are the Vaidya and the Tolman-Bondi metrics. The Vaidya metric describes the gravitational field associated with the eikonal approximation of an isotropic flow of unpolarized radiation, or, in other words, it represents a null fluid. It is usually employed in modeling the external field of radiating stars and evaporating black holes [6]. On the other hand, the Tolman-Bondi metric gives the gravitational field associated with dust matter and is frequently applied either in cosmological models or in describing the collapse of a star into a black hole [7]. Tolman-Bondi spacetimes embody the Schwarzschild solution, the Friedman universes, and the Oppenheimer-Snyder collapse, as well as inhomogeneous expansions and collapses.

At first sight these two metrics are completely different. Do the naked singularities that form in the collapse of null radiation and in the collapse of dust bear any relation with each other? Are there any features common to both solutions? And if this is the case what are the implications for cosmic censorship? Prior studies within the Tolman-Bondi class only focused on the so-called marginally bound and time-symmetric (bound) collapses. However, if some relation is to be found between the Vaidya and the Tolman-Bondi metrics one has to analyze the unbound case [8].

I find here that the naked singularities which appear in Vaidya and Tolman-Bondi spacetimes are of the same nature. In fact it is shown that various important features such as the degree of inhomogeneity of the collapse necessary to produce a naked singularity, the Cauchy horizon equation, the apparent horizon equation, the strength of the singularity, and the stability of the spacetime have a mutual correspondence in both metrics; i.e., I unify both metrics. For cosmic censorship, this result implies that if the shell-focusing singularities arising from the collapse of a null fluid are not artifacts of some (eikonal) approximation then the shell-focusing singularities arising from the collapse of dust are also not artifacts (and vice versa). Conversely, if the naked singularities are artifacts in one of them so are they in the other.

To set up the problem one describes the null radiation by the four-velocity null vector field k^a and the density ρ_r. The metric for spherically symmetric collapse of imploding radiation is then the Vaidya metric,

$$ds^2=-\left(1-\frac{2m(v)}{r}\right)dv^2+2dv\,dr+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\,, \tag{1}$$

where v is an advanced time and $k_a \equiv \partial_a v$. The function $m(v)$ gives the effective gravitational mass. To avoid shell crossing one requires that $m(v)$ be a nondecreasing function of v. The energy-momentum tensor associated with Eq. (1) is

$$T_{ab} = \frac{1}{4\pi r^2}\frac{dm}{dv} k_a k_b ,$$

which makes obvious the identification $dm/dv = 4\pi r^2 \rho_r$.

Dust matter is described by the timelike four-velocity vector field u^a of the particles and by the density ρ_m. The solution for spherical symmetric dust is in comoving coordinates given by the Tolman-Bondi metric,

$$ds^2 = -dt^2 + e^{2\omega(R,t)} dR^2 + r^2(R,t)(d\theta^2 + \sin^2\theta d\phi^2) . \tag{2}$$

Here R and t are, respectively, the comoving radial coordinate and the proper time of each concentric shell and $u_a \equiv \partial_a t$. The energy-momentum tensor for dust is $T_{ab} = \rho_m u_a u_b$ and Einstein field equations $G_{ab} = 8\pi T_{ab}$ yield

$$e^{\omega} = [1/\sqrt{1+2E(R)}] r' , \tag{3}$$

$$\tfrac{1}{2} \dot{r}^2 = E(R) + m(R)/r , \tag{4}$$

where primes and dots denote differentiation with respect to R and t, respectively. The function $m(R)$ is the effective gravitational mass within R related to $\rho_m(R,t)$ by $m' = 4\pi r^2 r' \rho_m$. One can always put $m = R$. The other function of integration $E(R)$ is the binding energy per unit mass of shell R and in order that metric (2) be nonsingular one takes $-\frac{1}{2} < E < \infty$. Bound objects are limited by $-\frac{1}{2} < E < 0$ while unbound objects have $E > 0$, $E = 0$ being the marginally bound case. One is interested in the limit $E \to \infty$. When $E > 0$ the solution of (4) for imploding matter is [9]

$$r = \frac{m}{E}\sinh^2\eta, \quad \sinh 2\eta - 2\eta = \frac{(2E)^{3/2}}{m}[t_c(R) - t] , \tag{5}$$

where η is an auxiliary parameter and $t_c(R)$ is another function of integration which defines the time that the shell with comoving coordinate R collapses to the singularity. In order to avoid shell crossing it is necessary to impose $t_c' \geq 0$ and $E' \leq 0$. According to Eq. (5) the evolution of $r(R,t)$ in the $E \to \infty$ regime satisfies

$$r = (2E)^{1/2}(t_c - t) + \frac{m}{2E}\ln\left[\frac{2(2E)^{3/2}}{m}(t_c - t)\right] + O(E^{-2}) \tag{6}$$

to the leading orders of E.

One now restricts the problem to self-similar collapses and links the most important features that are relevant to the appearance of naked singularities in the two models. The assumption of self-similarity is sometimes used to simplify the equations. Although precise geometric criteria for the development of naked strong singularities have been given only in the case of self-similar space times, some other examples not involving self-similarity also generate strong curvature singularities [10]. Therefore the assumption of self-similarity is not crucial. In the end the self-similar constraint is dropped and it is shown that the relation between these features is a more general fact. Self-similarity implies $m(v) = v/\mu$ and $t_c(m) = Bm$, where in this latter case the coordinate freedom to scale $R = m$ was used. The constants μ and B give the measure of the inhomogeneity of the collapse. For large μ and B one has highly inhomogeneous collapses where the outer shells collapse much later than the central ones.

In the Vaidya model a singularity first appears, signaled by the blowing up of the Kretschmann scalar, at $(v,r) = (0,0)$. A future Cauchy horizon, i.e., a naked singularity, appears [11] if $\mu \geq \mu_c = 16$, where a subscript c from here onwards denotes the critical value. The Cauchy horizon is the first outgoing null geodesic coming out of the singularity. When $\mu = \mu_c$ the equation for the Cauchy horizon is $v_{\rm CH} = 4r$ and for the apparent horizon is $v_{\rm AH} = 8r$. On the other hand, in the Tolman-Bondi metric a singularity forms at coordinates $(t,m) = (0,0)$ and a Cauchy horizon appears when $B \geq B_c = 16\sqrt{2E}$. For $B = B_c$ the Cauchy horizon evolves as

$$\left(\frac{m}{t}\right)_{\rm CH} = \frac{1}{16\sqrt{2E}}\left[1 + \frac{1}{8E} + O(E^{-2})\right]$$

and the apparent horizon as

$$\left(\frac{m}{t}\right)_{\rm AH} = \frac{1}{16\sqrt{2E}}\left[1 + \frac{1}{16E} + O(E^{-2})\right] .$$

In both metrics the apparent horizon always appears after the Cauchy horizon. The global nakedness of the singularity can be seen by making a junction onto the Schwarzschild spacetime.

The strength of the singularity is an important issue. There have been attempts to relate it to the stability problem. A singularity is said to be strong if an infinitesimal test body is crushed to zero proper volume as it approaches the singularity, i.e., if the body is destroyed at the singularity. This is associated with the so-called strong limiting focusing condition which states that the singularity is strong if $\psi \equiv \lim_{\lambda \to 0} \lambda^2 G_{ab} k^a k^b \neq 0$, where k^a is the null geodesic along the Cauchy horizon parametrized by λ [12]. In the Vaidya self-similar collapse one has $\psi = 8/\mu$. Hence for $\mu = \mu_c$ one has $\psi_{\rm V} = \frac{1}{2}$. In the Tolman-Bondi metric the measure of the strength is $\psi = 2C/(1+C)^2$, where now

$$C = 1 - \left(\frac{Bm}{t}\right)_{\rm CH}\left[1 - \frac{1}{(2E)^{3/2}B}\left(\frac{Bm/t}{Bm/t - 1}\right)^2 + O(E^{-2})\right]_{\rm CH} .$$

For $B=B_c$ one has $C=1+O(E^{-1})$; thus $\psi_{\rm TB}=\frac{1}{2}+O(E^{-1})$.

Another important point concerns the stability of the solutions that develop shell-focusing singularities with the formation of the associated Cauchy horizon. This is not a settled issue but there are some grounds to accept that in self-similar collapses the Cauchy horizon is stable at least to blueshift perturbations. The stability analysis against the blueshift criterion is motivated by the Kerr-Newman family which develops such a type of instability. If one injects massless fields along the future Cauchy horizon in the high-frequency approximation one has that the frequency shift of the radiation is given by $\nu_o/\nu_e=(u_ak^a)_o/(u_ak^a)_e$, where ν_o and ν_e are the observed and emitted frequency, respectively, and u^a is the four-velocity of the infalling matter at observation or emission [13]. In the Vaidya metric one may take, for $\mu=\mu_c$, $u^a=(0,-1,0,0)$ and $k^a=(1/2r)(4,1,0,0)$, so that $\nu_o/\nu_e=r_e/r_o=v_e/v_o$. In the Tolman-Bondi collapse if one takes

$$u^a=(1,0,0,0),\quad k^a=\frac{1}{(1+C)t^C}\left[1,\frac{1}{16\sqrt{2E}},0,0\right],$$

one obtains $\nu_o/\nu_e=(t_e/t_o)^C=(R_e/R_o)^C$, with C as given above. Since $\nu_o/\nu_e\le 1$ the future Cauchy horizons of both metrics are stable against the blueshift instability (although other modes can set in).

These results show that the scalars which appear in both metrics, namely, $\mu_c, B_c/\sqrt{2E}$ and $\psi_{\rm V},\psi_{\rm TB}$, have the same value and that the stability criterion is satisfied by both metrics. Since scalars are the quantities with an invariant meaning, this indicates that the Vaidya metric and the Tolman-Bondi metric in the $E\to\infty$ limit represent the same collapse.

Indeed, metric (2) uses comoving R rather than r as its radial coordinate so it is advisable to change to r. Now $dr=r'dR+\dot{r}\,dt$ and hence $(r'dR)^2=dr^2-2\dot{r}\,dr\,dt+\dot{r}^2dt^2$. Thus using Eqs. (3) and (6) and taking only the leading terms in E yields for the metric (2) the following:

$$ds^2=-\left[1-\frac{2m}{(2E)^{1/2}(t_c-t)}\right]\frac{dt^2}{2E}+\frac{2}{(2E)^{1/2}}\left[1-\frac{1}{2E}+\frac{m}{(2E)^{3/2}(t_c-t)}\right]dr\,dt+\frac{dr^2}{2E}+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,. \qquad (7)$$

Defining then the new coordinate $v\equiv t/\sqrt{2E}+r/2E=t_c(R)/\sqrt{2E}$ and taking the limit $E^{-1}\to 0$ in Eq. (7) gives the Vaidya metric (1). Here the fact was used that R is a (nondecreasing) function of m, and hence $v=t_c(m)/\sqrt{2E}$, which in turn can be inverted to give $m(v)$, the mass as a nondecreasing function of v. Notice that the Tolman-Bondi representation uses the time function $t_c(m)$, which is interpreted as the time that the shell with mass m collapses to the center, while the Vaidya representation prefers to use the mass function $m(v)$, which has the meaning of the mass accreted at time v. In the $E\to\infty$ limit the energy density of the matter in the Tolman-Bondi metric and the energy density of the radiation in the Vaidya metric are related by $\rho_r=2E\rho_m$.

Thus the Vaidya metric belongs to the Tolman-Bondi family. While the former provides only one function $m(v)$, the latter yields the two functions $E(R)$ and $t_c(R)$. The Tolman-Bondi metric is far-reaching enough to give continuously bound, marginally bound, and unbound collapses. The most unbound case yields the Vaidya metric. So one expects that major features which might arise in one of the metrics will also appear in the other. One example is the result that the strength in the Vaidya metric depends on the direction from which the geodesics enter the singularity. Within this perspective, this is a rediscovery of the same directional property found in the Friedman models [14]. Null fluids are, in principle, easier to treat than matter fields. A null fluid is the eikonal approximation of a massless scalar field. Thus if one shows that the naked singularities arising in the Vaidya metric can be derived from more fundamental (massless) fields, then the naked singularities which form in the Tolman-Bondi collapse may also be derived from more fundamental (massive) fields. The same types of relations and conclusions hold for charged radiation and charged dust matter.

This work is supported in part by a grant of CNPq, Brazil.

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