Current Carrying States in the Lowest Landau Level

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We have performed numerical calculations for noninteracting electrons in two dimensions in the high-magnetic-field (lowest-Landau-level) limit in the presence of a random potential. By identifying the first Chern character of each eigenstate, we show directly from the data for sizes varying from $\mathcal{N}=8$ flux quanta to $\mathcal{N}=128$ that the lowest Landau level has extended eigenstates only at a single energy E_c (the center of the band). The localization length ξ is found to diverge as $(E-E_c)^{-\nu}$, with $\nu=2.4\pm0.1$, in agreement with previous calculations.

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The prediction of the absence of truly extended states for noninteracting electrons in a disordered two-dimensional system in zero magnetic field [1] was followed only the following year by the discovery of the integer quantum Hall effect [2]. Halperin [3] showed that the latter implied that extended states must exist (or the localization length must diverge) at least at isolated energies for electrons in a two-dimensional system in this high-field (Landau-level) limit. The simplest two generic possibilities which could occur in the lowest Landau band are schematically shown in Fig. 1—(a) extended states occupy a finite region in the center of the energy spectrum, resulting in localization length ξ diverging at both the mobility edges; and (b) there is a singular point E_c away from which the localization length is finite.

Early work by Ono [4] based on a self-consistent Born approximation suggested an exponential divergence of localization length $\xi=\exp[E_0^2/(E-E_c)^2]$ at the center of the Landau band for the quantum problem. However, in the classical limit of a weak but smoothly varying potential, Trugman [5] showed that the problem maps onto a two-dimensional percolation problem. Here again there is a unique critical energy E_c at the center of the band, but in contrast to Ono's result, the localization length diverges as a power law $\xi\sim|E-E_c|^{-\nu_{cl}}$ with $\nu_{cl}=\frac{4}{3}$. [The classical limit can be formally defined by the limits $V(\mathbf{r})/\hbar\omega_c\to 0$, $l|\nabla V(\mathbf{r})|/V\to 0$, where V is the potential, $\hbar\omega_c$ is the cyclotron energy, and l is the cyclotron radius.] Since then, various numerical approaches based on finite-size scaling of localization length ξ [6–8] and Hall conductivity σ_{xy} [9] as well as on the series-expansion approach [10,11] for the quantum problem have been consistent with a single critical energy and a power-law divergence of ξ, as in the classical case, but with a different exponent ν_Q between 2.0 and 2.5. The most extensive numerical work due to Huckestein and Kramer [7] yields $\nu_Q=2.34\pm0.04$, in remarkable agreement with the analytic result $\nu_Q=\frac{7}{3}$ of Mil'nikov and Sokolov [12], which takes into account the effect of quantum tunneling in the percolation picture and is claimed to be correct in an intermediate limit.

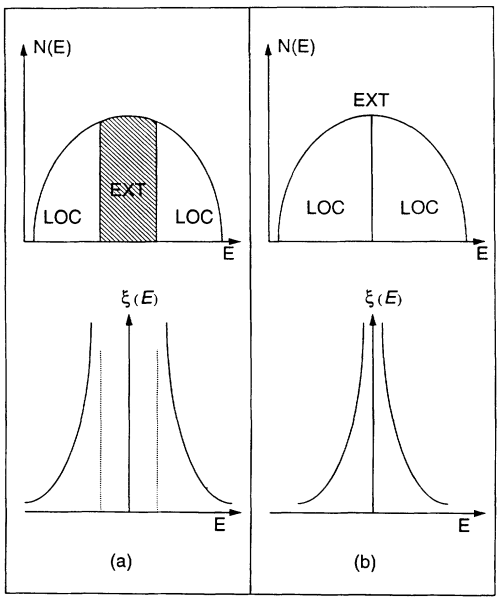

FIG. 1. Schematic drawing of the density of one-electron states and localization length for the two scenarios in the lowest Landau level with a random potential. In (a), extended states exist in a band near the center, and localized states in the band edges. In (b), states are localized at all energies, except at the center of the band, where the localization length diverges.

While these approaches have made a convincing case for scenario (b), this result is based on the observation that better fits can be obtained to the numerical results with that assumption. Consequently, it would be desirable to obtain this result more directly from numerical studies. Arovas *et al.* [13] have shown that one can use the topological properties of the eigenstates to distinguish between localized and extended states, which presents the rather unique opportunity of studying the evolution of extended states directly in a series of finite-size samples. In this paper, we have used this approach, and, by studying the variation with size, we are able to extrapolate to the thermodynamic limit. The existence of a topological invariant which is related to the current carrying capacity of the wave function offers a more direct method of identifying extended states than the conventional probes of the wave function [7] which typically require very large system size.

We consider the Hamiltonian $H=H_0+V(\mathbf{r})$ in two dimensions, with kinetic energy $H_0=(2m)^{-1}(\mathbf{P}+e\mathbf{A}/c)^2$. For a free particle, H_0 is a harmonic oscillator in the cyclotron coordinates, with a spectrum $E_n=(n+\frac{1}{2})\hbar\omega_c$ and a natural cyclotron frequency $\omega_c=eB/mc$. Each Landau level is extensively degenerate, with a density of states per unit area $\mathcal{N}/A=1/2\pi l^2$, where $l=(\hbar c/eB)^{1/2}$ is the magnetic length. In order to treat finite systems, we impose generalized periodic boundary conditions on a square of side L containing an integral number of flux quanta $\mathcal{N}=L^2/2\pi l^2$, by requiring $t(\mathbf{L}_j)|\Psi\rangle=\exp(i\theta_j)|\Psi\rangle$ $(j=1,2)$, where $t(\mathbf{r})$ is the magnetic translation operator [14]. We consider a number of different random potentials $V(\mathbf{r})$ described below. The condition of high magnetic field is imposed by projecting onto the lowest Landau level.

The Hall conductivity σ_{xy} can be calculated as a function of both energy E and boundary-condition angles θ_1 and θ_2. The Kubo formula for the Hall conductivity for noninteracting electrons may be written as a sum over the occupied eigenstates [15]:

$$\sigma_{xy}(E;\theta_1,\theta_2)=\sum_{m=1}^{\mathcal{N}}\delta\sigma_{xy}(m;\theta_1,\theta_2)\Theta(E-E_m)\,, \qquad (1)$$

where

$$\delta\sigma_{xy}(m;\theta_1,\theta_2)=\frac{ie^2}{\hbar}\left(\frac{\partial}{\partial\theta_1}\langle m|\frac{\partial}{\partial\theta_2}|m\rangle-\frac{\partial}{\partial\theta_2}\langle m|\frac{\partial}{\partial\theta_1}|m\rangle\right) \qquad (2)$$

is the Hall conductivity of the mth eigenstate $|m\rangle$ with energy E_m, and Θ is a step function. As demonstrated by Thouless and co-workers [15,16] and others [17–19], the unweighted average of $\delta\sigma_{xy}(m)$ over all boundary angles is necessarily an integral multiple of e^2/h. This integer can be identified as $C_1(m)$, the first Chern index, which is a topological invariant characterizing the topological properties of the wave functions [20].

Since $C_1(m)$ is the average of the Hall conductivity of the state, the Chern index is clearly a measure of the current carrying capacity of an eigenstate, which in turn is a statement about its extensiveness. In fact, Arovas *et al.* [13] showed that there is a simple connection between delocalization of wave functions and their topological characteristic which leads to a nonzero σ_{xy}. The wave function of state which carries an average nonzero Hall current can be forced to vanish at any specified position in real space by the appropriate choice of boundary conditions whereas that of a state with zero average Hall conductivity cannot. This means that the first Chern index $C_1(m)$ tells us the sensitivity of nodes of the wave function to changes in boundary conditions. This property can be used to differentiate between states with $C_1(m)\neq 0$ whose nodes can be moved at will (extended) and those with $C_1(m)=0$ whose nodes are confined in space (localized). In this paper we adopt the above definition of extended and localized states and perform numerical calculation over finite-size samples to study the statistical properties of these extended states.

We have studied systems with sizes $\mathcal{N}=8$, 16, 32, 64, and 128, keeping the magnetic field fixed, so that scaling with $\mathcal{N}$ is the same as scaling with the area of the sample. For each size we have diagonalized the system between 100 and 2000 random potential configurations depending on size. We have considered different types of random potentials. Numerically, the simplest to implement is the Gaussian white-noise potential, where the system is converted into a $M\times M$ grid, in which the microscopic length $a_0=L/M$ is small compared to l. The potential is generated by choosing an independent Gaussian-distributed random value for each plaquette. However, in order to check for universality, we have also considered a potential due to randomly placed short-range scatterers which may be closer to the experimental situation. For this case, we randomly sprinkle the $M\times M$ grid with $N_{\rm imp}$ unit scatterers with a Gaussian potential

$$v(\mathbf{r})=\pm v_0\exp(-r^2/2\sigma^2)\,,$$

half of which are positive and half negative. In either case, the $L\times L$ sample is periodically extended to tile the entire plane. In contrast to the classical percolation model, which corresponds to a smooth and long-range correlated potential, we have kept $\sigma/l=0.5$ and $a_0/l<1$ in both cases, so that our potential has only short-range correlations and we are dealing with the quantum limit. For a system projected onto the lowest Landau level there is electron-hole symmetry for the ensemble average. Consequently, for every given random potential we always keep another copy with the sign of the potential reversed at every point in space in the ensemble average. The density of states averaged over our finite ensemble is thus a symmetric function of energy.

For each size and potential we calculate the Chern index of each eigenstate; from the ensemble average, we obtain the average density of states and density of extended states. We find that the extended states cluster around the center of the Landau band. We calculate both the average number $\mathcal{N}_c$ of extended states (states with nonzero Chern number) as well as the mean width of the band of extended states

$$\Delta E=\left[\sum_n(E_n-\bar{E})^2/\mathcal{N}_c\mathcal{N}_s\right]^{1/2}=\left[\sum_n E_n^2/\mathcal{N}_c\mathcal{N}_s\right]^{1/2}, \qquad (3)$$

where the sum is over the extended states of all $\mathcal{N}_s$ samples for a given size. The latter equality follows from the fact that the mean energy for the ensemble is guaranteed to be zero by electron-hole symmetry.

In Fig. 2 we show the total density of states as well as the density of extended states for the Gaussian random potential for system sizes $\mathcal{N}=8$ and $\mathcal{N}=128$. As can be

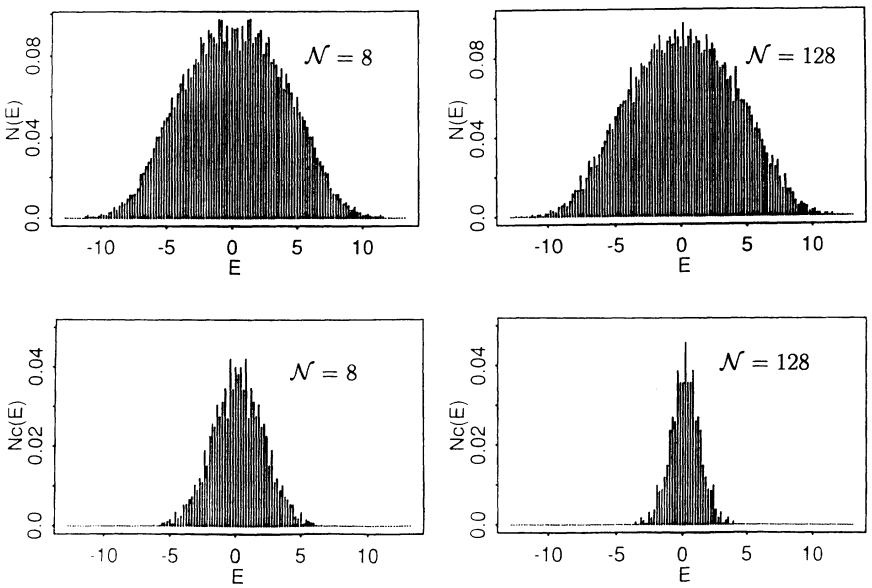

FIG. 2. Density of states of all states $N(E)$, as well as the density of nonzero Chern number states $N_c(E)$ for samples with flux quanta $\mathcal{N}$ equal to 8 and 128.

seen clearly, the total density of states does not change much with $\mathcal{N}$. In contrast, the band of the current carrying (extended) states becomes markedly narrower for the larger size.

Figure 3 shows our results for the variation of ΔE with system size, normalized to E_0, the total width of the lowest Landau band defined as in (3) but with all $\mathcal{N}\mathcal{N}_s$ states included. For both random potentials, our data fall on straight lines, suggesting a universal power-law variation of the width of extended state bands with system size:

$$\Delta E \sim \mathcal{N}^{-x},$$

with

$$x = 0.21 \pm 0.01 .$$

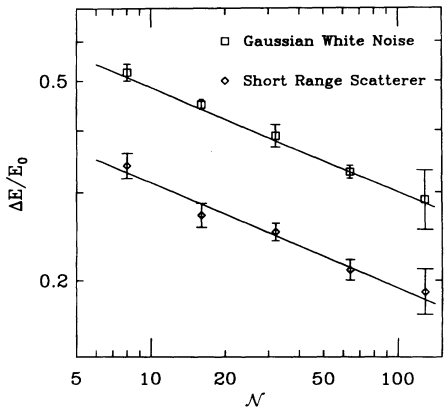

FIG. 3. Relative width of extended state band $\Delta E/E_0$ vs $\mathcal{N}$ for two different kinds of random potentials (see text) on a log-log scale. Solid lines are linear least-squares fits to the data.

This demonstrates directly that $\Delta E \to 0$ as $\mathcal{N} \to \infty$; i.e., as we extrapolate to the thermodynamic limit, all extended states collapse to a single energy [scenario (b) in Fig. 1], in agreement with results obtained less directly by other methods.

From the exponent x, we can obtain the correlation-length exponent ν_Q ($\xi \sim |E - E_c|^{-\nu_Q}$) from the following observation. For a finite-size sample with length scale L, since states with localization length $\xi > L$ appear extended, one expects $L \sim (\Delta E)^{-\nu_Q}$, or $\Delta E \sim \mathcal{N}^{-1/2\nu_Q}$, so $x = (2\nu_Q)^{-1}$, which gives $\nu_Q = 2.4 \pm 0.1$, in agreement with previous results [6–12].

In Fig. 4 we show how the average number of extended

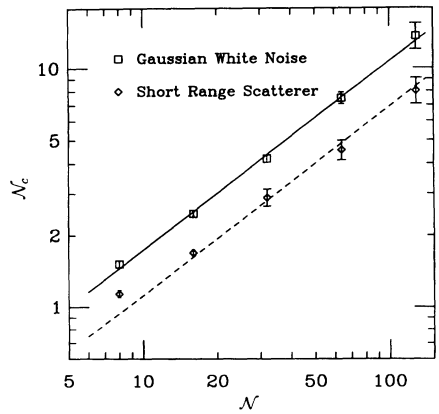

FIG. 4. Log-log plot of the number of extended states $\mathcal{N}_c$ vs $\mathcal{N}$ for the two random potentials. The solid line is the linear least-squares fit to the data of Gaussian white-noise potential. The dashed line demonstrates that the data for short-range scatterers are consistent with a straight line with the same slope, although there is measurable deviation for small sizes.

states $\mathcal{N}_c$ scales with $\mathcal{N}$ for both potentials. For the Gaussian white-noise potential we have $\mathcal{N}_c \sim \mathcal{N}^y$ with $y = 0.79 \pm 0.02$, which gives within the statistical errors $y = 1 - x$, as would be expected if the extended states were to have a finite density of states at E_c (the center of the band) in the thermodynamic limit [21]. For the randomly positioned impurity potential we find a small measurable curvature on the double logarithmic plot, but the data are consistent with the same y as in the white-noise potential in the large-$\mathcal{N}$ limit, as shown by the dashed line in Fig. 4.

In conclusion, by adopting a definition of extended states based on a topological invariant of the eigenstates and performing finite-size scaling, we are able to show directly that all extended states collapse to a single energy level E_c (the band center for electron-hole symmetric case) in the thermodynamic limit, with a finite density of extended states at the band center. In addition, the localization length ξ is found to diverge in a power-law fashion $\xi \sim |E - E_c|^{-\nu_Q}$ with $\nu_Q = 2.4 \pm 0.1$, in agreement with more conventional measures of the wave functions.

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[1] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[2] K. von Klitzing, G. Ebert, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[3] B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).

[4] Y. Ono, J. Phys. Soc. Jpn. **51**, 2055 (1982).

[5] S. A. Trugman, Phys. Rev. B **27**, 7539 (1983).

[6] H. Aoki and T. Ando, Phys. Rev. Lett. **54**, 831 (1985); T. Ando and H. Aoki, J. Phys. Soc. Jpn. **54**, 2238 (1985).

[7] B. Huckestein and B. Kramer, Phys. Rev. Lett. **64**, 1437 (1990).

[8] J. T. Chalker and P. D. Coddington, J. Phys. C **21**, 2665 (1988).

[9] R. N. Bhatt, Bull. Am. Phys. Soc. **32**, 623 (1987); D. P. Arovas, R. N. Bhatt, and P. B. Littlewood (unpublished); see also Ref. [22].

[10] S. Hikami, Phys. Rev. B **29**, 3726 (1984); S. Hikami and E. Brezin, J. Phys. (Paris) **46**, 2021 (1985); S. Hikami, Prog. Theor. Phys. **76**, 1210 (1986).

[11] R. R. P. Singh and S. Chakravarty, Nucl. Phys. **B265 [FS15]**, 265 (1986); M. P. Gelfand and S. Chakravarty, Bull. Am. Phys. Soc. **36**, 598 (1991).

[12] G. V. Mil'nikov and I. M. Sokolov, Pis'ma Zh. Eksp. Teor. Fiz. **48**, 494 (1988) [JETP Lett. **48**, 536 (1988)].

[13] D. P. Arovas, R. N. Bhatt, F. D. M. Haldane, P. B. Littlewood, and R. Rammal, Phys. Rev. Lett. **60**, 619 (1988).

[14] For details of the approach, see F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).

[15] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. Den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

[16] Q. Niu, D. J. Thouless, and Y. S. Wu, Phys. Rev. B **31**, 3372 (1985).

[17] J. E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983).

[18] M. Kohmoto, Ann. Phys. (N.Y.) **160**, 343 (1985).

[19] H. Aoki and T. Ando, Phys. Rev. Lett. **57**, 3093 (1986).

[20] For a review, see E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, 1991), Chap. 9.

[21] R. N. Bhatt, in *Physical Phenomena at High Magnetic Fields,* Proceedings of the National High Magnetic Field Laboratory Conference, Tallahassee, Florida, edited by E. Manousakis *et al.* (Addison-Wesley, Reading, 1991), p. 65. The preliminary data shown in that talk using smaller sizes and fewer samples suggested a vanishing density of extended states. This is because for the low density of random scatterers studied there, singular behavior in $N(E)$ appears (see Fig. 6 in that paper), which is not present in the current data. This will be clarified in a longer paper.