Existence and Stability of Semilocal Strings

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It has recently been shown that a certain model field theory [which gives the Abelian Higgs model a global SU(2) symmetry by duplicating the scalar sector] possesses static finite-energy vortex solutions despite having a simply connected vacuum manifold. It is demonstrated here that they are stable or unstable according to whether m_s is less than or greater than m_v (where m_s and m_v are the masses of the Higgs and vector particles, respectively). At the boundary, $m_s = m_v$, there is a two-parameter family of solutions all saturating a Bogomol'nyi bound. The relationship of the vortices to CP^n σ-model lumps is pointed out.

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Topological defects are well known to exist in theories with spontaneously broken symmetries, whether global or local [1]. We are motivated to study such objects by their ability to survive for long periods after the phase transition by which they are produced in the very early Universe [2]. In particular, local and global vortices [3], global monopoles [4], and global texture [5] are all thought to be able to produce effects observable in the Universe today, and thereby hold the potential for providing direct information about physics at the unification scale.

Recently, Vachaspati and Achúcarro [6] have examined a simple field-theory model which combines global and local symmetries in an interesting way. It consists of a complex doublet of scalar fields Φ with only the overall phase gauged. The Lagrangian is therefore

$$\mathcal{L} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - \tfrac{1}{2}\lambda(\Phi^\dagger\Phi - \eta^2)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad (1)$$

where $D_\mu = \partial_\mu - ieA_\mu$. The symmetries of this "extended" Abelian Higgs model are a global SU(2) and a local U(1). The potential has an even larger symmetry: The real and imaginary parts of the two components of Φ (call them ϕ_1 and ϕ_2) form a vector under an O(4) group of transformations. A natural generalization is to make Φ an N-vector of SU(N), in which case the symmetry group of the potential is O($2N$). At zero temperature the scalar field has an expectation value $\|\Phi\| = \eta$, and the vacuum manifold M is isomorphic to $O(2N)/O(2N-1) \simeq S^{2N-1}$. Vachaspati and Achúcarro showed [6] for $N=2$ that vortex solutions exist, which they termed "semilocal" strings, although the usual topological condition on M, that it be multiply connected, is not satisfied. This raises the question of whether these solutions are in fact stable.

This Letter is concerned with examining the stability of these solutions. Numerical methods give very clear evidence that when $\beta \equiv \lambda/e^2 < 1$ the vortices are stable. This stability disappears when $\beta > 1$, and the magnetic flux in the vortex tends to spread out to infinity. Perhaps the most interesting case is $\beta = 1$, where it is possible to prove that there is a two-parameter family of solutions saturating the Bogomol'nyi [7] bound on the energy $E = 2\pi\eta^2$, which have values of $\|\Phi\|$ at the vortex core between 0 and η. There is a close similarity between these vortices and the instantons in CP^{N-1} σ models [8]. The radial dependence of the magnetic field departs radically from the usual $e^{-m_v r}$ behavior found in Nielsen-Olesen vortices [9] and becomes a power law r_0^2/r^4, with r_0 arbitrary. Thus it would appear that although there is a Higgs mechanism operating to give the vector boson a mass, the magnetic fields cease to be confined as β becomes greater than 1. Natural units $\hbar = c = 1$ shall be used throughout, while i and j shall take values in {1,2} when used as subscripts or superscripts.

The problem is to assess the stability of the two-dimensional static finite-energy solutions to the equations of motion resulting from the Lagrangian (1). This means minimizing the energy functional

$$\mathcal{E} = \int d^2x [|D_i\Phi|^2 + \tfrac{1}{2}\lambda(\Phi^\dagger\Phi - \eta^2)^2 + \tfrac{1}{2}B^2], \qquad (2)$$

where $B = \epsilon^{ij}\partial_i A_j$. The finite-energy condition means that every term in (2) must vanish as $r \to \infty$. The vanishing of the potential term requires that the image of the circle at spatial infinity Φ_∞ must lie in M. When M is not simply connected it is clear from the continuity of the field that for those maps Φ whose $r \to \infty$ limit is not contractible in M there must be some region in the plane where $\Phi(x^i)$ leaves M. Here there is energy density associated with an object usually described as a vortex or a string. In the present case, $\pi_1(M)$ vanishes so the argument just given fails. However, for finite energy the covariant derivative term in (2) must vanish at infinity as well, which means that Φ_∞ is just a gauge transformation:

$$\Phi_\infty(\theta) = \exp\left(ier\int_0^\theta d\theta' A'_\theta\right)\Phi_\infty(0). \qquad (3)$$

The continuity of Φ tells us that its phase change at infinity is $2\pi n$ with n an integer. This integer is a topological invariant of finite-energy field configurations, and it measures the number of times that Φ_∞ winds around the gauge orbit which passes through $\Phi_\infty(0)$.

Following Bogomol'nyi [7] and Vachaspati and Achúcarro [6], Eq. (2) can be rewritten in an enlightening way. After an integration by parts we find

$$\mathcal{E}=2\pi|n|\eta^2+\int d^2x\{\tfrac{1}{2}|D_i\Phi\pm i\epsilon_{ij}D^j\Phi|^2+\tfrac{1}{2}[B\pm e(\Phi^\dagger\Phi-\eta^2)]^2+\tfrac{1}{2}(\beta-1)e^2(\Phi^\dagger\Phi-\eta^2)^2\}\,, \qquad (4)$$

where the signs are taken according to the sign of n. Thus in the special case $\beta=1$, the energy is minimized at the value $2\pi|n|\eta^2$ when the fields satisfy the first-order equations

$$(D_i\pm i\epsilon_{ij}D^j)\Phi=0,\quad B\pm e(\Phi^\dagger\Phi-\eta^2)=0\,. \qquad (5)$$

When $\beta>1$ (<1), this value forms a lower (upper) bound, since the last term is positive (negative) semidefinite.

Vachaspati and Achúcarro showed that there are solutions of the form $\Phi=f(\xi)e^{i\theta}\Phi_1$, where $\xi^i=e\eta x^i$ are dimensionless coordinates, and $\xi=|\xi^i|$. In this case the field equations reduce to exactly those for the ordinary Nielsen-Olesen vortex [9]. They did not, however, check the stability of these solutions to small perturbations. The most general one-vortex ansatz which maintains the expected cylindrical symmetry is

$$\Phi=f(\xi)e^{i\theta}\Phi_1+g(\xi)e^{im\theta}\Phi_2,\quad A_i=\epsilon_{ij}\xi^j a(\xi)/e\xi\,, \qquad (6)$$

with $||\Phi_1||=\eta=||\Phi_2||$ and $\Phi_1^\dagger\Phi_2=0$. The orthogonality of Φ_1 and Φ_2 ensures that the effect of a spatial rotation can be removed from both components by a suitable $SU(2)\times U(1)$ symmetry transformation. We shall see that it is sufficient to examine the $m=0$ case only, and so with ansatz (6) the energy functional becomes

$$\mathcal{E}=2\pi\eta^2\int_0^\infty\xi\left[(f')^2+(g')^2+\frac{1}{2\xi^2}(a')^2+\frac{(1-a)^2}{\xi^2}f^2+\frac{a^2}{\xi^2}g^2+\frac{1}{2}\beta(f^2+g^2-1)^2\right]d\xi\,, \qquad (7)$$

while the Bogomol'nyi equations are

$$f'+[(a-1)/\xi]f=0,\quad g'+(a/\xi)g=0\,,$$
$$a'+\xi(f^2+g^2-1)=0\,. \qquad (8)$$

Let us first consider the solutions away from the Bogomol'nyi limit of $\beta=1$. The problem is to minimize (7) subject to the following boundary conditions:

$$f=0,\quad g'=0,\quad a=0,\quad \text{at } r=0\,,$$
$$f\to1,\quad g\to0,\quad a\to1,\quad \text{as } r\to\infty\,. \qquad (9)$$

If $g_0\equiv g(0)\neq0$ then we depart from the usual Nielsen-Olesen vortex solution. Qualitatively, it would seem that for large β, there should be a significant decrease in the energy if $g\neq0$ in the core of the string where f departs from 1. Conversely, as $\beta\to0$, the cost in gradient energy might seem to outweigh any small energy reduction afforded by a nonzero g. In fact, it is possible to show that when $\beta>1$, there are no minimum-energy vortices of finite core radius. The proof proceeds by constructing a one-parameter family of field configurations whose energy tends to the Bogomol'nyi limit as the value of the parameter is taken to infinity. The field configurations are

$$f=\frac{\xi}{\xi_0}\left(1+\frac{\xi^2}{\xi_0^2}\right)^{-1/2},\quad g=\left(1+\frac{\xi^2}{\xi_0^2}\right)^{-1/2},$$
$$a=\frac{\xi^2}{\xi_0^2}\left(1+\frac{\xi^2}{\xi_0^2}\right)^{-1}, \qquad (10)$$

for which the energy is $E=2\pi\eta^2(1+1/3\xi_0^2)$. Thus as $\xi_0\to\infty$, E tends to the Bogomol'nyi bound. Any truly stable solution therefore must saturate this bound, and we see from (4) that when $\beta>1$ saturation can only happen if $B=0$ everywhere. This is inconsistent with the total flux being $2\pi/e$. The possibility remains of a metastable solution, but numerical work described next seems to preclude it.

I have investigated the minima of (7) using a relaxation method described in Ref. [10]. I have also checked the stability of the Nielsen-Olesen-type solutions to small perturbations in g, by looking numerically for negative-eigenvalue solutions to the Schrödinger-like equation

$$\left[-\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\right)+\frac{a^2}{\xi^2}+\beta(f^2-1)\right]\psi=\omega^2\psi\,, \qquad (11)$$

where ϕ is a small perturbation around $g=0$. For reasons of space I will merely state some results, and details of the method will be presented elsewhere [11]. For $\beta=100$, 30, 10, and 3, I found negative eigenvalues, which approached zero from below as β decreased towards unity. For $\beta=1$, 0.3, 0.1, 0.03, and 0.01 no negative eigenvalues were found. For perturbations with angular quantum number m [see Eq. (6)], the second term in (11) is replaced by $(a-m)^2/\xi^2$. Note then that the "potential" term in (11) for $m>1$ is everywhere larger than that for $m=1$, and therefore if the lowest $m=1$ eigenvalue is positive then so are all $m>1$ eigenvalues. The stability of the solution towards $m=1$ perturbations is guaranteed, since small fluctuations in f have an eigenvalue equation identical to (11). It follows that checking the stability of the solution towards $m=0$ perturbations is sufficient for assuring its complete stability.

It would therefore appear that the Nielsen-Olesen-type vortices are stable in this model only if $\beta<1$. When

$\beta > 1$ they are unstable towards forming a condensate in the core, which seems to be able to spread out to infinity. A definite statement cannot be given on this point, because there may be a metastable solution with finite core radius. However, using the relaxation code, I was unable to find one which converged convincingly.

We now turn to the special case $\beta = 1$, where minimizing E is equivalent to solving the Bogomol'nyi equations (8). Any solution to these equations satisfying the correct boundary conditions (9), and this includes the Nielsen-Olesen-type string, is guaranteed to be at an absolute minimum of the energy, and therefore there can be no perturbations with negative eigenvalues. However, there is a zero-eigenvalue solution, which is

$$\psi = \psi_0 \exp\left[-\int_0^{\xi} a(\xi')\xi'^{-1} d\xi' \right].$$

This indicates that there is a degeneracy in the solutions to the Bogomol'nyi equations, for this zero mode is present for any $g(\xi)$ that is a solution, not just $g = 0$. It remains to show that such solutions exist.

The problem with Eqs. (8) is that they are nonautonomous, and therefore many of the useful theorems about first-order ordinary differential equations do not hold, in particular each point (f,g,a) does not lie on a unique trajectory [12]. It is therefore convenient to change these equations into autonomous ones. One helpful point to note is that the equations for f and g are not independent. In fact, g may be replaced by $f\xi_0/\xi$, reducing the three equations (8) to two. To make them autonomous we define a new variable $\tau = \tan^{-1}(\xi/\xi_0)$, a new function $\rho = (f^2 + g^2)^{1/2}$, and finally another function z whose solution is $\sin\tau$. This brings the equations to the form

$$\frac{d\rho}{d\tau} = \rho \frac{z^2 - a}{z(1-z^2)^{1/2}}, \quad \frac{da}{d\tau} = \xi_0^2 \frac{z(1-\rho^2)}{(1-z^2)^{3/2}},$$
$$\frac{dz}{d\tau} = (1-z^2)^{1/2}. \tag{12}$$

It is required to show that there exists a solution to these equations which starts at point $p = (\rho_0, 0, 0)$ and passes through $q = (1,1,1)$ for any ξ_0 and some $\rho_0 \in (0,1)$. As far as the physical solutions to these equations are concerned, the region of interest is the open cube C defined by $\rho, a, z \in (0,1)$, and its surface Σ. The proof proceeds by showing that all paths X_{ρ_0} starting at p must leave the cube through a certain subset $S \subset \Sigma$ which includes the point a. The geometry of S is such that removing q divides it into two disjoint pieces S_1 and $S_2 \backslash q$. One then shows that according to whether ρ_0 is near 0 or 1, $X_{\rho_0} \cap \Sigma$ is in different pieces. The continuity of X_{ρ_0} then guarantees that there exists some $\rho_0 \in (0,1)$ for which X_{ρ_0} includes q. Note that q is a critical point.

We first define S. Since z' and a'' are both positive at p, the path must commence by entering the cube C. Furthermore, a' and z' are positive everywhere in C, so X_{ρ_0} cannot pass through either of the faces $z = 0$ or $a = 0$. Neither can it reach $z = 1$, for a' and ρ' diverge there. The paths therefore must either disappear to infinity or reach the point q. Two other regions are now excluded: $\rho = 1$ with $a > z^2$ and $\rho = 0$ with $a < z^2$, because ρ' is directed into the cube. The region $\rho = 0$, $a \geq z^2$ can also be excluded by examining the behavior of trajectories when ρ is small. For these there is an approximate solution for a, namely, $a(\tau) \simeq \frac{1}{2}\xi_0^2 \tan^2\tau + A$, where A is an integration constant. Thus ρ can be solved for:

$$\rho = \bar{\rho} \sec\tau (\cot\tau)^A \exp(-\tfrac{1}{4}\xi_0^2 \tan^2\tau). \tag{13}$$

Trajectories therefore approach $\rho = 0$ but cannot reach it. As $\rho_0 \to 0$ these approximate solutions hold good everywhere and A vanishes. It follows also that for sufficiently small ρ_0, the path leaves C through the face $a = 1$ at $\tan\tau = \sqrt{2}/\xi_0$ or $\rho = \rho_0(1 + 2/\xi_0^2)^{1/2} \exp[-(1 + \xi_0^2)/2]$, which is in the region $S_1 = \{\rho, a, z : \rho, z \in (0,1), a = 1\}$. It remains to show that for ρ_0 sufficiently close to 1 the path exits somewhere in the region $S_2 = \{\rho, a, z : \rho = 1, z \in (0,1], a \leq z^2\}$. We proceed by expanding around $\tau = 0$, near which point

$$z \simeq \tau, \quad a(\tau) \simeq \tfrac{1}{2}\xi_0^2(1-\rho_0^2)\tau^2,$$
$$\rho \simeq \rho_0 \{1 + \tfrac{1}{2}[1 - \tfrac{1}{2}\xi_0^2(1-\rho_0^2)]\tau^2\}. \tag{14}$$

Thus for ρ_0 close to 1 the path reaches $\rho = 1$ and leaves C at $\tau^2 \simeq 2(1-\rho_0)/\rho_0$. Now, it is well known that paths which are solutions of autonomous ordinary differential equations cannot cross or end, except at critical points [12]. Therefore, as ρ_0 is increased away from small values, the line defined by the points X_{ρ_0} must pass smoothly from S_1 into S_2 as a function of ρ_0. The only way that this is possible while remaining in $S_1 \cup S_2$ is through the critical point q. We have now shown that there exists a solution to the Bogomol'nyi equations (8) with $\rho_0 \neq 0$ for any $\xi_0 > 0$. Furthermore, since Φ_2 has an arbitrary phase α once Φ_1 is fixed, vortex solutions are actually labeled by a complex number [13] (the special case $\xi_0 = 0$ corresponds to the Nielsen-Olesen-type vortex, for which solutions are already known to exist). Note also that if the inequality $\rho_0^2 > 1 - 2/\xi_0^2$ is not satisfied, ρ' is negative everywhere in C, since a is always greater than z^2. Thus any path that reaches q must satisfy this inequality, which shows that as $\xi_0 \to \infty$, $\rho_0 \to 1$. Thus in this limit the scalar field remains in the vacuum manifold everywhere. To summarize, when $\beta = 1$ there is a family of vortex solutions labeled by a complex number $\xi_0 e^{i\alpha}$, all of which have energy $2\pi\eta^2$.

The asymptotic behavior of these solutions is very different from that of the Nielsen-Olesen vortex, for which [9] $f(\xi) \simeq 1 - c_1 \xi^{-1/2} \exp(-\sqrt{2\beta}\xi)$ and $a(\xi) \simeq 1 - c_2 \xi^{1/2} \exp(-\sqrt{2}\xi)$. Up to and including terms of fourth degree in $\chi \equiv \xi_0/\xi$, it is found for the vortex in the extended Abelian Higgs models that at large distances,

$$f \simeq 1 - \tfrac{1}{2}\chi^2 + (\tfrac{3}{8} - \xi_0^{-2})\chi^4, \quad g \simeq \chi(1 - \tfrac{1}{2}\chi^2), \\ a \simeq 1 - \chi^2 + (1 - 4\xi_0^{-2})\chi^4. \tag{15}$$

The resulting power-law decrease in the magnetic field at infinity is a significant departure from the usual exponential decay associated with the confinement of magnetic flux. Furthermore, the width of the flux tube is completely undetermined, instead of being the Compton wavelength of the vector particle.

In this respect, the long-distance properties of the vortices in extended Abelian Higgs models with N scalar fields are closely similar to those of instantons in the two-dimensional CP^{N-1} σ model [8]. The CP^{N-1} model is a theory of an N-dimensional complex scalar field $\mathbf{n}$ constrained to have unit norm, whose Lagrangian is simply $\mathcal{L} = |D_\mu \mathbf{n}|^2$, where $D_\mu = \partial_\mu - iA_\mu$. The vector field A_μ has no derivative terms and acts purely to give the theory a local U(1) gauge invariance. Solving for A_μ one finds $A_\mu = i\mathbf{n}^* \cdot \partial_\mu \mathbf{n}$. It is well known that in two dimensions the instantons of the CP^{N-1} σ model have arbitrary size [8]. Defining a complex coordinate $z = x + iy$, the general CP^{N-1} instanton is

$$\mathbf{n} = \frac{\mathbf{u}(z - z_0)/w + \mathbf{v}}{(1 + |z - z_0|^2/|w|^2)^{1/2}}, \tag{16}$$

where $|\mathbf{u}|^2 = 1 = |\mathbf{v}|^2$, $\mathbf{u}^* \cdot \mathbf{v} = 0$, and w is an arbitrary scale parameter. Note that as $|z - z_0| \to \infty$, $\mathbf{n} \to e^{i \arg z}\mathbf{u}$. Furthermore $|\mathbf{u}^* \cdot \mathbf{n}|$ and $|\mathbf{v}^* \cdot \mathbf{n}|$ are equal to f and g, respectively in Eqs. (10) and so the kinship to the vortex is clear. The existence of instantons in 2D CP^{N-1} σ models arises from the nontriviality of π_2 of the target manifold. In three space dimensions this implies the existence of a sort of global monopole in the extended Abelian Higgs models, but a rather unusual one, because the target manifold is constructed by identifying the orbits of the U(1) gauge group. Thus the images of nontrivial mappings from the 2-plane into CP^{N-1} are disks whose boundaries lie on a U(1) gauge orbit. This means that global monopoles must be the termini of gauge vortices. When $N = 2$ the σ model is equivalent to the O(3) σ model [8], since $CP^1 \simeq S^2$, and a species of global texture can be expected since $\pi_3(S^2) \simeq Z$. Details will be given in a separate publication [11].

In conclusion, this combination of local and global topological defects may find application in a cosmological context, where strings, global monopoles, and global texture all provide promising theories of structure formation [3-5]. A model which combines all three may prove irresistible.

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