Hyperbolic Evolution System for Numerical Relativity

C. Bona and J. Massó

Departament de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca, Spain

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Einstein evolution equations are written as a hyperbolic system of balance laws. A harmonic time coordinate is used with zero shift vector (harmonic slicing). The principal part of the evolution system reduces to a set of uncoupled wave equations in first-order form. The relevance for three-dimensional numerical relativity of both the harmonic slicing and the resulting evolution system is stressed.

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In this paper, we write down the evolution system of Einstein field equations as a hyperbolic first-order system of balance laws in terms of densities, fluxes, and sources. This is a structure well known to those who numerically solve the Navier-Stokes equations for hydrodynamics and this fact should provide numerical relativists with the opportunity to employ many of the powerful methods of computational fluid dynamics in simulations of general relativistic systems.

The numerical techniques so far used in general relativity have been developed in a somewhat *ad hoc* manner simply because the Einstein equations have not been put in the form of hyperbolic balance law equations. Therefore a large body of numerical methods useful for such equations were not applicable to general relativity. The results of this paper should change that situation and hopefully will stimulate rapid progress in the development of numerical simulations of relativistic astrophysical systems.

The main concern in numerical relativity is to follow the space-time evolution from given initial data. This sets a strong connection with the general relativistic Cauchy problem: One can expect to be able to develop consistent stable numerical algorithms only when the existence and uniqueness of the solution is ensured by a well-posed initial-value problem. As is well known, the Einstein field equations must be supplemented with some coordinate conditions and this determines the mathematical structure of the evolution system.

Existence and uniqueness problems have been proven for hyperbolic systems and these theorems have been applied to the Einstein field equations with different coordinate conditions which lead to a hyperbolic evolution system. The most popular ones are the harmonic coordinates:

$$\Box x^a = 0 \quad (a=0,1,2,3), \tag{1}$$

where the symbol $\Box$ stands for the d'Alembert differential operator acting on functions, namely,

$$\Box f \equiv g^{ab} \partial^2_{ab} f - \Gamma^c \partial_c f, \tag{2}$$

and we have used

$$\Gamma^a \equiv g^{bc} \Gamma^a_{bc} = -\Box x^a, \tag{3}$$

where Γ^a_{bc} are the connection coefficients associated with the space-time metric g_{ab}. These conditions have recently been generalized to the case where the Γ^a are given in terms of suitably regular conditions [1] (but not necessarily equal to zero).

Other authors impose the harmonicity conditions (1) by using the d'Alembert operator (2) associated with some background metric [2], or keep only the harmonicity condition for the time coordinate,

$$\Gamma^0 = -\Box x^0 = 0, \tag{4}$$

and choose the space coordinates such that

$$g^{0i} = 0 \quad (i=1,2,3), \tag{5}$$

leading to the "harmonic slicing" [3].

Harmonic slicing.— Allowing for (5), the line element can be decomposed in the form

$$ds^2 = -\alpha^2 dt^2 + g_{ij} dx^i dx^j \tag{6}$$

and this means that the time lines are orthogonal to the t = const slices (zero shift vector). This ensures that the congruence of temporal lines will be as regular as the time slicing itself.

This is contrary to what happens when using conditions (1): The time slicing would be the same and the temporal lines would be given by

$$\Box x^i = 0 \quad (i=1,2,3), \tag{7}$$

but, as is well known, the d'Alembert operator does not preserve the causal character of the solutions, so that coordinate horizons could appear in the course of evolution. We conclude that the simpler condition (5) is safer than (7) for numerical applications.

The time slicing is governed by the lapse function α in (6). An algebraic condition on the lapse is obtained by writing Eq. (4) in the form

$$\partial_t(\alpha/\sqrt{g}) = 0, \tag{8}$$

so that

$$\alpha = C(x)\sqrt{g}, \tag{9}$$

where g stands for the determinant of the three-dimensional metric g_{ij} and $C(x)$ is an integration constant [4].

Equation (9) shows that the time evolution will slow down whenever the space volume element $\sqrt{g}$ decreases and this is the key for showing the singularity avoidance properties of the harmonic slicing, as we did in a previous work [5]. Singularity avoidance is an important requirement for numerical applications and it also has been shown for the popular "maximal slicing" condition [5,6]. However, the evolution system in the maximal slicing case is not hyperbolic and, to our knowledge, existence and uniqueness theorems are yet to be proven in that gauge.

Note that singularity avoidance does not mean absence of coordinate singularities, as is shown by the following example:

$$ds^2 = e^{-2t}(-dt^2+dx^2)+dy^2+dz^2, \qquad (10)$$

which is a Kasner-like form of the Minkowski metric; a coordinate singularity appears at a finite value of the proper time which is avoided by the harmonic slicing (it does not appear at any finite value of the coordinate time). The unphysical slicing is produced by an unphysical choice of initial conditions (the time derivative of the vacuum metric is chosen to be spatially constant).

Hyperbolic evolution system.—The Einstein field equations can be written [7] as

$$\tfrac{1}{2}(\Box g^{ab}+\partial^a\Gamma^b+\partial^b\Gamma^a)-\Gamma^{acd}\Gamma^b_{cd}=R^{ab}, \qquad (11)$$

and the evolution system is constructed from the spatial components of (11). In the harmonic slicing, it can be written as a first-order system of balance laws [8] in the quantities

$$Q^{ij}\equiv\partial_t g^{ij},\quad D_k^{ij}\equiv\partial_k g^{ij} \qquad (12)$$

as follows:

$$\partial_t[(\sqrt{g}/\alpha)Q^{ij}]-\partial_k[\alpha\sqrt{g}(D^{kij}+g^{ki}\Gamma^j+g^{kj}\Gamma^i)]=(\sqrt{g}/\alpha)Q^{ik}Q_k^j-2\alpha\sqrt{g}[R^{ij}+\Gamma^{ikl}\Gamma^j_{kl}+L^iL^j-\Gamma^i\Gamma^j],$$
$$\partial_t[D_k^{ij}]-\partial_k[Q^{ij}]=0, \qquad (13)$$

where we have used

$$L^i\equiv\partial^i\ln\alpha,$$
$$\Gamma^i\equiv-\Box x^i=\tfrac{1}{2}g_{jk}D^{ijk}-D_k^{ki}-L^i. \qquad (14)$$

One could complete the system (13) with the coordinate condition (8) and its spatial derivatives

$$\partial_t L_i+\tfrac{1}{2}\partial_i Q_k^k=0 \qquad (15)$$

to evolve α and L^i, respectively. The complete system [(13), (8), and (15)] has the balance law structure [9]

$$\partial_t D(u)+\partial_k F^k(u)=S(u), \qquad (16)$$

where the densities D, fluxes F^k, and sources S are vector-valued functions of the set of variables

$$u=(\alpha,L^i,g^{ij},Q^{ij},D_k^{ij}). \qquad (17)$$

The form (16) of the evolution system is well known in hydrodynamics and standard numerical methods for this kind of system have been developed. However, the consistency and stability of all such methods have been established only for hyperbolic systems, that is, systems in which the characteristic matrix (the Jacobi matrix of the projection of the fluxes along a given direction) has real eigenvalues and a full set of independent eigenvectors. Actually, the most powerful methods do use the spectral decomposition of the characteristic matrix to obtain a finite difference version of the equations. And the complete evolution system for (17) is not hyperbolic [10].

A hyperbolic first-order system in $\partial_t Q^{ij}$, $\partial_t D^{kij}$ can be obtained [3] by taking the time derivative of (13) and noticing that the mixed space-time components of (11) (the "momentum constraints" in our gauge) can be written as

$$\partial_t\Gamma^i=-2\alpha^2R^{0i}+Q_k^kL^i-2Q_j^iL^j+\Gamma^i_{jk}Q^{jk}, \qquad (18)$$

so that it can be used to eliminate the Γ^i terms in the principal part of the derived system.

We shall follow a much simpler way by considering (13) as an evolution system for the quantities

$$u=(\alpha,\Gamma^i,g^{ij},Q^{ij},D_k^{ij}) \qquad (19)$$

[note that Γ^i has replaced L^i in (17)] and completing it with (8) and (18) [instead of (15), which is no longer used] so that L^i is obtained algebraically from Γ^i by using (14). The complete system [(13), (8), and (18)] in the quantities (19) is hyperbolic as we will show in what follows.

Transport terms.—To study the hyperbolicity of a system of the form (16) we need to consider only the principal part (densities and fluxes). This is closely related to the "operator splitting" approach in numerical applications, in which the time evolution is decomposed into a combination of two kinds of processes, namely, (i) transport, governed by the flux-conservative law

$$\partial_t D(u)+\partial_k F^k(u)=0, \qquad (20)$$

and (ii) evolution, driven by the source terms only.

The hyperbolicity of the system can ensure the stability of the numerical methods used in the transport step only. The stability of the whole scheme would require in addition that of the system of nonlinear ordinary differential equations given by the source terms. This is beyond the scope of this work, where we will be concerned with the transport step only.

Going back to our evolution system, let us note that the ten quantities α, Γ^i, and g^{ij} are constant in the transport step (they do not have flux terms) and that the evolution of the remaining 24 quantities can then be written as

$$\partial_t[(\sqrt{g}/\alpha)Q^{ij}] - \partial_k[\alpha\sqrt{g}(g^{kl}\lambda_l^{ij})] = 0\,,$$
$$\partial_t[\lambda_k^{ij}] - \partial_k[Q^{ij}] = 0\,, \qquad (21)$$

where we have used

$$\lambda_k^{ij} \equiv D_k^{ij} + \delta_k^i \Gamma^j + \delta_k^j \Gamma^i\,. \qquad (22)$$

The interest in this change of variables (22) resides in the fact that the resulting system (21) consists of six uncoupled identical subsystems of four linear equations in Q^{ij}, λ_k^{ij} (with i,j fixed). This is important if one is planning to apply either spectral or implicit finite difference methods in the transport step, because it greatly reduces the dimensions of the matrices involved.

But the main point is that every subsystem in (21) has the structure of the wave equation in first-order form. This shows that the evolution system is hyperbolic with characteristic surfaces given by the light cones (plus the time lines if one takes into account the quantities which are constant in the transport step), providing a good starting point for applying standard existence and uniqueness theorems. This also means that the arsenal of numerical methods designed and tested for the three-dimensional wave equations is at our disposal: This is a fast way of coming in on three-dimensional numerical relativity. We have actually applied standard finite difference explicit methods in that way to vacuum metrics of the form (10) with excellent results [11].

Let us finally remark that the quantities λ_k^{ij} could be introduced from the beginning [at the price of adding source terms to the second equation in (13)]. The time components of (11),

$$\Box g^{00} = 2R^{00} + \Gamma^{0cd}\Gamma_{cd}^{0}\,, \qquad (23)$$

could have been used instead of (8) and (18) to follow the evolution of the lapse function and its derivatives, leading to a complete evolution system consisting of seven (instead of six) uncoupled wave equations, which will be then hyperbolic. This would be similar to the "relaxed system" that one constructs in the standard harmonic gauge. We think that it is worth investigating the eventual advantages of this alternative approach to see whether or not they compensate both the introduction of the additional wave equation in the transport step as well as the replacement in the source step of the eighteen sourceless (nonevolving) quantities D_k^{ij} by λ_k^{ij}.

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[3] Y. Choquet-Bruhat and T. Ruggeri, Commun. Math. Phys. **89**, 269 (1983). In that work, the term "algebraic gauge" was used. We will avoid using this term because it may be a little too general. A large number of algebraic gauges have already been introduced in numerical relativity. For this reason we have used the more specific term "harmonic slicing" to emphasize that the time coordinate is harmonic in that gauge.

[4] One could be tempted to reduce the number of dynamical variables by replacing everywhere the lapse function α. This does not seem to be a good idea, because the auxiliary function $C(x)$ and its derivatives would appear explicitly in the principal part of the system and, therefore, a high degree of smoothness in $C(x)$ would be required for applying standard theorems.

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[9] The relevance of that structure in numerical relativity has been discussed in Ref. [8], where a different, nonhyperbolic, evolution system was presented in that form.

[10] The eigenvalues are real but there is not a full set of independent eigenvectors. Some authors would say that the system is hyperbolic but not "strictly hyperbolic."

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