Berry's Phase, Motive Forces, and Mesoscopic Conductivity

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A ring in a magnetic field whose direction varies in space is considered. It is shown that the Berry phase accumulated by the spins of electrons encircling the ring affects the conductance of the ring in a way similar to the Aharonov-Bohm effect. A time-dependent Berry phase is shown to induce a motive force in the ring. This motive force couples to the electron's spin, similar to the way Faraday's law couples an electromotive force to the electric charge.

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Since its discovery in 1983, Berry's phase [1] has been the subject of numerous investigations [2]. The simplest example that illustrates the concept of Berry's phase is that of a spin $\frac{1}{2}$ that follows adiabatically a magnetic field whose direction varies in time. When the magnetic field returns to its initial direction, the spin wave function is found to have acquired a geometric phase factor. This phase can be regarded as induced by a geometric flux, similar to the phase shift induced by an electromagnetic flux in the Aharonov-Bohm effect [1,2].

Motivated by this similarity between the fluxes, this paper investigates Berry-phase analogies to two physical effects involving an electromagnetic flux: the induction of current in a conducting ring by a time-dependent electromagnetic flux (through Faraday's law), and the effect of time-independent flux on the conductivity of a mesoscopic ring (through the Aharonov-Bohm effect) [3,4]. In these analogies, the electron's spin plays the role played by the electric charge in the electromagnetic effects. Another analogy, introducing persistent currents induced by Berry's phase in ballistic rings, was recently discovered in an instructive work by Loss, Goldbart, and Balatsky [5], who have also conjectured related effects of Berry's phase on the magnetoconductance. In the spirit of that work, we define here a thought experiment in which electrons in a mesoscopic conducting ring follow adiabatically a magnetic field whose direction varies spatially, and thus accumulate Berry's phase. By mapping that phase onto an effective vector potential, we show that when the phase is time independent, it affects the ring's conductance. When the phase varies in time, it creates a (nonelectro) motive force that induces a current in the ring. By discussing the analogies to the electromagnetic phenomena, we point out that the effect of a time-independent geometric flux is a nonlocal effect and is therefore observable only in mesoscopic rings, while the effect of a time-dependent geometric flux should be observed also in macroscopic rings, i.e., *it does not depend on phase coherence.* We examine the conditions for adiabaticity to be maintained, and their dependence on the disorder in the ring. It should be stressed that similar to the persistent currents discussed by Loss, Goldbart, and Balatsky [5], the phenomena we discuss are independent of the electric charge. Our results stem from the Zeeman interaction, and are therefore valid for all spin-$\frac{1}{2}$ particles, irrespective of their charge.

We consider a quasi-one-dimensional ring, whose radius is a. The ring lies in the x-y plane, and its center is in the origin. A nonuniform magnetic field is applied on the ring in the following way: First, a magnetic field B_ϕ tangent to the ring is induced by a current-carrying wire lying along the z axis. Second, a uniform field B_z is applied on the system, parallel to the z axis. Adopting a cylindrical coordinate system, the total magnetic field has a component B_ϕ created by the wire in the $\hat{\boldsymbol{\phi}}$ direction, and a component B_z in the $\hat{\mathbf{z}}$ direction. Along the ring, the magnitude of the field is constant, but the direction varies. In fact, it follows a cone-shaped path, where the angle between the cone and the z axis, denoted by α, satisfies $\tan\alpha = B_\phi/B_z$ (see Fig. 1). The spin of an elec-

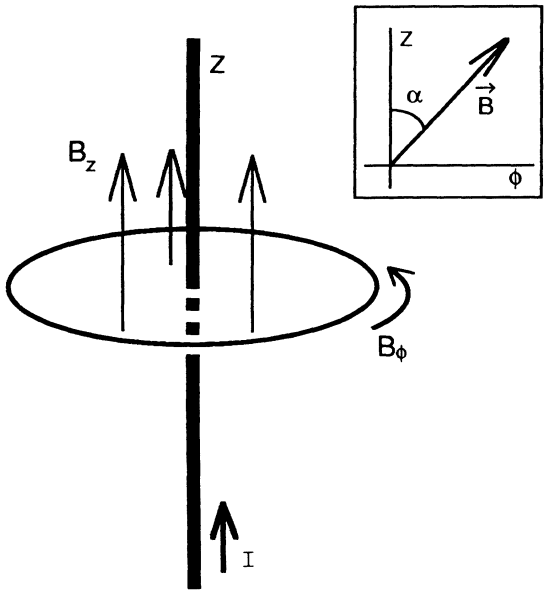

FIG. 1. The physical problem considered. A ring is put in a uniform external magnetic field B_z and a tangential magnetic field B_ϕ created by the current-carrying wire. The ratio between the two fields defines the angle α.

tron that slowly encircles the ring is then expected to follow the direction of the magnetic field and thus accumulate a geometric phase of

$$\Omega_g^{\uparrow(\downarrow)} \equiv \pi(1 \pm \cos\alpha) , \tag{1}$$

i.e., half the solid angle subtended by the magnetic field it goes through. (The $\uparrow, +$ and $\downarrow, -$ refer to the spin being parallel and antiparallel to the field, respectively [1]). The angle α is determined by the current through the wire and by the uniform field along the $\hat{\mathbf{z}}$ direction.

Our discussion of the above-described thought experiment involves three parts. In the first part we use the Born-Oppenheimer approach in order to separate the Hamiltonian of the system into adiabatic and nonadiabatic parts. We show that the adiabatic part includes a geometric vector potential that couples to the electron's spin. Assuming that the ring is one dimensional, its Hamiltonian is

$$H = \Pi^2/2M + V(\phi) - \mu \mathbf{B}(\phi)\cdot\boldsymbol{\sigma} , \tag{2}$$

where $\Pi = -(i/a)d/d\phi - eB_z\pi a/2c$ is the generalized momentum (a system of units where $\hbar = 1$ is utilized), $V(\phi)$ is the impurity potential along the ring, μ is the magnetic moment, M is the mass of an electron, and $\boldsymbol{\sigma}$ is the Pauli matrix vector. Attempting to discuss the adiabatic limit, we diagonalize the spin-dependent part of the Hamiltonian, treating the angle ϕ as a parameter. We obtain the eigenstates

$$|\uparrow(\phi)\rangle = \begin{pmatrix} i\cos\frac{1}{2}\alpha\, e^{-i\phi} \\ -\sin\frac{1}{2}\alpha \end{pmatrix}, \quad |\downarrow(\phi)\rangle = \begin{pmatrix} i\sin\frac{1}{2}\alpha\, e^{-i\phi} \\ \cos\frac{1}{2}\alpha \end{pmatrix},$$

corresponding to the spin being parallel (antiparallel) to the magnetic field. The corresponding eigenvalues are $\mp\mu B$, where $B \equiv (B_\phi^2 + B_z^2)^{1/2}$. Defining $|\phi\rangle$ as the eigenstate of the operator $e^{i\hat{\phi}}$, the two sets of states $\{|\uparrow(\phi)\rangle \otimes |\phi\rangle | 0 \le \phi < 2\pi\}$ and $\{|\downarrow(\phi)\rangle \otimes |\phi\rangle | 0 \le \phi < 2\pi\}$ constitute together a basis of the Hilbert space of the Hamiltonian (2). Each one of these sets spans a subspace in which the spin is either parallel or antiparallel to the magnetic field. The Hamiltonian H is now written as a sum of an adiabatic part H_0, whose matrix elements are nonzero only within each subspace, and a purely nonadiabatic part H_1, whose matrix elements are nonzero only between states of different subspaces. Using the method outlined recently by Aharonov *et al.* [6] we find that

$$H_0 = \frac{[\Pi - A_g]^2}{2M} + V(\phi) - \mu\mathbf{B}(\phi)\cdot\boldsymbol{\sigma} + \frac{1}{8Ma^2}\sin^2\alpha$$

and

$$H_1 = (1/2M)[(\Pi - A_g)A_g + A_g(\Pi - A_g)] ,$$

where

$$A_g = (1/2a)\sin\alpha[\cos\alpha\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\phi}} - \sin\alpha\,\sigma_z] .$$

By construction, H_0 has a set of eigenstates $|n,\uparrow\rangle = |\uparrow(\phi)\rangle \otimes \psi_n^{\uparrow}(\phi)$ in which the spin is parallel to the field, and a set of eigenstates $|n,\downarrow\rangle = |\downarrow(\phi)\rangle \otimes \psi_n^{\downarrow}(\phi)$ in which the spin is antiparallel to the field. The wave functions $\psi_n^{\uparrow}(\phi)$ and $\psi_n^{\downarrow}(\phi)$ are eigenstates of the Hamiltonians

$$H_0^{\uparrow(\downarrow)} \equiv \left\{ \frac{1}{2M}\left[\Pi - \frac{1}{2\pi a}\Omega_g^{\uparrow(\downarrow)}\right]^2 + V(\phi) \mp \mu B + \frac{1}{8Ma^2}\sin^2\alpha \right\}, \tag{3}$$

with corresponding eigenvalues $E_n^{\uparrow(\downarrow)}$. Each of these Hamiltonians [Eq. (3)] is a projection of the full Hamiltonian onto one of the subspaces defined above.

The adiabatic approximation neglects H_1. As seen in Eq. (3), when this approximation is valid the ring can be viewed as composed of two uncoupled electron gases. Those gases are subject to the effect of different geometric vector potentials, induced by the spatial variation of the direction of the magnetic field [7]. The magnitude of this vector potential is independent of the electric charge, but is rather determined by the direction of the spin being parallel or antiparallel to the magnetic field. The geometric vector potential is limited in magnitude, reflecting the restriction of the geometric phase to the range 0–2π. Thus, for spin-$\frac{1}{2}$ particles the geometric flux enclosed by the ring is limited to the order of one flux quantum. This restriction distinguishes the geometric flux from the electromagnetic one. The two electron gases are also subject to the effect of opposite constant potential energy, originating from the Zeeman interaction, to identical electromagnetic flux $B_z\pi a^2$, and to identical impurity potentials. These observations conclude the first part of our discussion.

In the second part of the discussion we analyze the conditions under which H_1 can be disregarded. We start the discussion by considering the ballistic case, where $V(\phi) = 0$, a case for which the full Hamiltonian (2) can be exactly diagonalized [5]. The exact eigenstates for this case are $e^{in\phi}[C_1|\uparrow(\phi)\rangle + C_2|\downarrow(\phi)\rangle]$, where n is an integer, $\binom{C_1}{C_2}$ is an eigenvector of the matrix

$$\begin{pmatrix} \frac{1}{2Ma^2}[n'^2 - (2n'-1)\cos^2\frac{1}{2}\alpha] - \mu B & -\frac{2n'-1}{4Ma^2}\sin\alpha \\ -\frac{2n'-1}{4Ma^2}\sin\alpha & \frac{1}{2Ma^2}[n'^2 - (2n'-1)\sin^2\frac{1}{2}\alpha] + \mu B \end{pmatrix}, \tag{4}$$

and $n' \equiv n - eB_z a^2/2c$. The energies corresponding to these eigenstates are the eigenvalues of the matrix (4). The adia-

batic approximation taken in the previous paragraphs amounts to approximating $C_1=1$, $C_2=0$ ($C_1=0$, $C_2=1$) for eigenstates in which the spin is parallel (antiparallel) to the magnetic field. As seen from the matrix (4), for a ballistic ring the adiabatic approximation is valid (for all values of α) when $\mu B \gg n'/Ma^2$, i.e., when the time it takes an electron whose velocity is n'/a to encircle the ring, Ma^2/n', is larger than the spin precession time $(\mu B)^{-1}$. Hence, for electrons at the Fermi level, the condition for the adiabatic approximation to hold is

$$\mu B a/v_F \gg 1 , \tag{5}$$

where v_F is the Fermi velocity. In the presence of impurity potential, we treat H_1 as a perturbation. The eigenstates of $H_0^{\uparrow(\downarrow)}$ are not eigenstates of the momentum operator Π, and therefore H_1 couples each eigenstate $|n,\uparrow\rangle$ to a quasicontinuum of states $|m,\downarrow\rangle$ (and vice versa). As a result of that coupling, each adiabatic eigenstate acquires a finite lifetime τ. This lifetime can be perturbatively evaluated using the diagrammatic impurity technique [4]. The details of this calculation are given elsewhere [8]. Up to leading order in $1/\epsilon_F\tau_{el}$ (where ϵ_F is the Fermi energy and τ_{el} is the elastic mean free time), the inverse lifetime of an adiabatic state at the Fermi energy is

$$\frac{1}{\tau}=\frac{2D}{a^2}\frac{\sin^2\alpha}{(2\mu B\tau_{el})^2+1} , \tag{6}$$

where D is the diffusion constant. For an electron to be affected by the geometric flux, its spin has to follow the direction of the magnetic field long enough so that the geometric phase it accumulates is significant. Hence, when the angle α is of order unity, the lifetime of the adiabatic states, given in Eq. (6), has to be longer than the typical time it takes a diffusing electron to encircle the ring, $(2\pi a)^2/D$. This condition is fulfilled when

$$2\mu B\tau_{el} \gg 1 . \tag{7}$$

Therefore, in the diffusive regime, the adiabatic approximation is valid when the spin precession time is much shorter than the time between elastic scattering events. Equation (7) can be shown to be the condition for adiabaticity also when $\alpha \ll 1$ [8], a case in which the electron has to encircle the ring many times in order to accumulate a significant geometric phase. The adiabatic condition can be formulated in terms of the ac conductivity of the ring [8].

We conclude the second part of our discussion by making a few comments regarding the adiabatic condition (7). First, we interpret its physical origin. As argued by Thouless [9], the single-electron eigenstates in a disordered system are a superposition of plane waves, with typical spread of $\hbar/l$, where l is the elastic mean free path. In kinetic energy terms, this width is translated into $\hbar/\tau_{el}$. Therefore, the matrix elements of the generalized momentum operator Π between states whose kinetic energies differ by more than $\hbar/\tau_{el}$ are negligible. Consequently, so also are the matrix elements of H_1. On the other hand, flips of the spin due to H_1 occur only at the Fermi level, i.e., between states whose kinetic energies differ by $2\mu B$. Hence, when condition (7) is valid, the nonadiabatic matrix elements between states at the Fermi level are negligible, and the lifetime becomes long. Second, we comment that one should distinguish here between the diffusive limit $\omega_c\tau_{el} \ll 1$ (where ω_c is the cyclotron frequency) and the Landau levels limit $\omega_c\tau_{el} \gg 1$. The relevant limit is determined by the value of the electron's g factor. Here we assume that the diffusive limit applies. Third, we comment on the relevance of $1/\tau$ to interference effects. As discussed above, the geometric phase accumulated by the electron depends on the direction of its spin. If that direction is flipped at various points along the path, this phase is randomized. Hence, nonadiabatic spin flips dephase the interference. In the present work we neglect all other mechanisms of dephasing, and therefore τ is to be identified with the phase-breaking time τ_ϕ. The ratio of the circumference of the ring to the phase-breaking length $L_\phi \equiv (D\tau_\phi)^{1/2}$, denoted by Γ, is then

$$\Gamma \equiv \frac{2\pi a}{L_\phi}=\frac{\sqrt{8}\pi\sin\alpha}{[1+(2\mu B\tau_{el})^2]^{1/2}} . \tag{8}$$

We emphasize that as long as no other dephasing mechanisms are present, this ratio depends neither on the radius a, nor on the temperature T. And finally, we note that for an elastic mean free time of 10^{-11} sec and a g factor of 10, the adiabaticity condition (7) is satisfied for fields larger than 0.1 T.

In the third part of the discussion we assume that the adiabatic limit is applicable and analyze the physical consequences of the geometric flux. Assuming that the Zeeman energy μB is smaller than the Fermi energy ϵ_F, our ring consists of the two uncoupled electron gases described above. The electric conductance of the ring is the sum of the conductances of the two gases. As discussed extensively in recent years [3,4], the conductance of a mesoscopic ring depends on the flux threading the ring, through the Aharonov-Bohm effect. In the configuration we discuss, the flux threading the sample is the sum of an electromagnetic flux $\phi_{em}=B_z\pi a^2$ and the geometric flux $\phi_g=\frac{1}{2}\phi_0(1\pm\cos\alpha)$, where the $\pm$ refers to electrons whose spin is parallel (antiparallel) to the field. It should be noted here that the sum of the two geometric fluxes corresponding to the two gases equals a flux quantum. Since all properties of the ring are periodic with respect to one flux quantum, one can view the two electron gases as subject to the influence of geometric fluxes of equal magnitude and opposite signs. For rings in the diffusive regime, the flux dependence of the conductance is manifested in two different contexts, namely, the average conductance of an ensemble of macroscopically identical rings and the sample-specific fluctuations. The flux-dependent part of the average conductance was calculated by Al'tshuler, Aronov, and Spivak [10]. Using their

formula for each of the two electron gases in our ring, we find that the total quantum correction to the conductivity is

$$\delta\sigma = -\frac{e^2 a}{\Gamma}\left[\frac{\sinh(\Gamma)}{\cosh(\Gamma)-\cos[4\pi(\phi_{em}+\phi_g)/\phi_0]}+\frac{\sinh(\Gamma)}{\cosh(\Gamma)-\cos[4\pi(\phi_{em}-\phi_g)/\phi_0]}\right]. \quad (9)$$

The effect of the geometric flux on the sample-specific fluctuations of the conductance is best understood when the periodicity of those oscillations with respect to B_z is considered. In the absence of geometric flux ($B_\phi=0$), the ϕ_0 flux periodicity yields a field periodicity of $\Delta B_z = \phi_0/\pi a^2$, irrespective of the spin direction. In the presence of geometric flux, a variation of B_z varies both the electromagnetic and the geometric fluxes. Thus the periodicity with respect to B_z is changed, and is no longer independent of the spin direction. Specifically, when $B_z \ll B_\phi$ (i.e., $\alpha \to \pi/2$), the geometric flux is approximately $\phi_0 B_z/2B_\phi$, and the B_z period becomes

$$\Delta B_z = \phi_0/\pi a^2 \pm \phi_0/2B_\phi, \quad (10)$$

where the + (−) sign refers to the spin being parallel (antiparallel) to the field.

Equations (9) and (10) summarize our predictions for the effect of Berry's phase on the conductivity of a mesoscopic ring. We now turn to discuss the case of a time-dependent geometric flux, and, in particular, the currents it induces in the ring. We consider the case in which the tangential magnetic field is $B_\phi = B_\phi^0 \cos\omega t$, and consequently the angle α becomes time dependent too. In order to avoid, at this stage, the complications involved in the analysis of the adiabatic condition for that case, we limit ourselves to the case in which $\epsilon_F + \omega \ll \mu B_z$, i.e., the electron gas is spin polarized, and absorption of an energy quantum $\hbar\omega$ still does not allow electrons to flip their spins. For semiconducting rings, this condition may be fulfilled at fields of the order of 1 T. Under this assumption, the electron gas in the ring is subject to the effect of a time-dependent geometric flux $\phi_g = \frac{1}{2}\phi_0[1+\cos\alpha(t)]$. Consequently, this gas is subject to a motive force ϵ, given by $\epsilon = -d\phi_g^{1(2)}/dt$, and this motive force induces a current in the ring, according to Ohm's law. Assuming that $B_\phi^0 \ll B_z$, the motive force induced by the time dependence of ϕ_g is

$$\epsilon = -\tfrac{1}{2}\omega\phi_0(B_\phi^0/B_z)^2 \sin 2\omega t. \quad (11)$$

For $B_z = 1$ T, $B_\phi = 0.2$ T, and $\omega = 1$ GHz, this motive force has an amplitude of 10^{-7} V.

There are a few points that should be stressed regarding the case of a time-dependent geometric flux. First, contrary to the effect of a time-independent flux, the time-dependent geometric flux exerts a force on the electron [11], similar to the electric force exerted by a time-dependent electromagnetic flux. Thus, similar to the observation of currents induced due to Faraday's law, the observation of currents induced by the geometric flux *does not depend on the electron phase being coherent along the ring.* Those currents should be observed in macroscopic rings as well as in mesoscopic ones. Second, the motive force induced in the ring is not electric, since if the electrons were replaced by neutrons, the picture would not have changed. The field, given by the derivative of the vector potential with respect to the time, does not couple to the electric charge, but rather to the direction of the spin. And finally, the origin of the motive force exerted on the electron can be understood by noting that in our symmetrical structure the sum of the orbital and spinor angular momenta in the z direction is time independent even when the angle α is time dependent. Thus, a change in α transfers angular momentum from the spin to the orbital motion of the electron. A more general analysis of this force, from points of view of both linear and angular momenta, will be presented in a future publication.

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