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# Controlling Chaos Using Time Delay Coordinates

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The Ott-Grebogi-Yorke control method is analyzed in the case that the attractor is reconstructed from a time series using time delay coordinates. It turns out that the control formula of Ott, Grebogi, and Yorke should be modified in order to apply to experimental systems if time delay coordinates are used. We reveal that the experimental surface of section map depends not only on the actual parameter but also on the preceding one. In order to meet this dependence two modifications are introduced which lead to a better performance of the control. To compare their control abilities they are applied to simulations of a Duffing oscillator.

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In 1990 Ott, Grebogi, and Yorke (OGY) [1] proposed a new method of controlling a chaotic dynamical system by stabilizing one of the many unstable periodic orbits embedded in a chaotic attractor, through only small time-dependent perturbations in some accessible system parameter. This makes OGY's approach quite different from other previously published methods on controlling chaos [2].

OGY's method has attracted the attention of many physicists interested in applications of nonlinear dynamics. One reason for this is that OGY stress that all values needed to achieve control can be obtained from an experimental signal starting with the well-known embedding technique [3,4]. Therefore the control method can in principle be applied to experimental systems where the dynamical equations are not known. Indeed, Ditto, Rauseo, and Spano demonstrated recently [5] a first control of a physical system using the method of Ott, Grebogi, and Yorke.

With regard to possible applications we investigate the OGY control method in the case that the attractor is reconstructed from a time series using time delay coordinates. It turns out that the control formula of OGY should be modified in order to apply to experimental systems if time delay coordinates are used. The main argument will be that during the control process one switches the control parameter $p$ from $p_{i-1}$ to $p_i$ at times $t_i$ ($t_i$ is the time of the $i$th piercing of the surface of section by the trajectory). But, if one uses delay coordinates, the experimental surface of section map $P$ does not only depend on the new actual parameter $p_i$ (as OGY implicitly assume) but also on the old one $p_{i-1}$. In order to meet this dependence two modifications of the control algorithm are proposed. Their control abilities are compared with the original OGY formula by applying them to a time series obtained from simulations of a Duffing oscillator.

Let us briefly recall the OGY control idea. For simplicity we restrict ourselves to a two-dimensional discrete dynamical system (e.g., the surface of section map $P$ of a three-dimensional continuous system). There also exist extensions of the method to higher-dimensional dynamical systems [6,7]. Let the system depend on some accessible parameter $p \in (p_0-\delta p_{\max}, p_0+\delta p_{\max})$ with maximal possible perturbation $\delta p_{\max}$, $\xi_{i+1}=P(\xi_i,p)$. Let $\xi_F=P(\xi_F,p_0)$ denote the unstable fixed point on the attractor which one wants to stabilize. The control idea is to monitor the system until it comes close to the desired fixed point and then change $p$ by a small amount such that the next state $\xi_{i+1}$ will fall into the stable direction of the fixed point. To do this one uses the first-order approximation of $P$ near $\xi_F$ and $p_0$,

$$\delta\xi_{i+1} \cong A\delta\xi_i + \mathbf{w}\delta p_i ,$$

with $\delta\xi_i=\xi_i-\xi_F$, $\delta p_i=p_i-p_0$, $A=D_\xi P(\xi_F,p_0)$, and $\mathbf{w}=\partial P/\partial p(\xi_F,p_0)$. Writing the linearization $A$ as $A=\lambda_u \mathbf{e}_u \mathbf{f}_u+\lambda_s \mathbf{e}_s \mathbf{f}_s$, with $\mathbf{e}_u$ ($\mathbf{e}_s$) the unstable (stable) eigendirections of $A$ with eigenvalues $\lambda_u$ ($\lambda_s$) and $\mathbf{f}_u$ ($\mathbf{f}_s$) their contravariant basis vectors, i.e., $\mathbf{f}_s\cdot\mathbf{e}_u=\mathbf{f}_u\cdot\mathbf{e}_s=0$ and $\mathbf{f}_s\cdot\mathbf{e}_s=\mathbf{f}_u\cdot\mathbf{e}_u=1$, the condition that $\xi_{i+1}$ falls on the local stable manifold of the fixed point can be formulated as $\mathbf{f}_u \cdot \delta\xi_{i+1}=0$, which yields the control formula [8] for the new value of the control parameter $p_i = p_0 + \delta p_i$,

$$\delta p_i = -(\lambda_u/\mathbf{f}_u \cdot \mathbf{w})\mathbf{f}_u \cdot \delta\xi_i \,. \tag{1}$$

The control is only activated if the resulting change in the parameter $\delta p_i$ is less than the maximal allowed disturbance $\delta p_{\max}$; otherwise $\delta p_i$ is set to zero.

Let us now consider the case that the only information about the system is obtained by some measurement process which is mathematically realized by some scalar function $Z$ on the state space $M$. If $Y(t) \in M$ is the state of the system at time $t$, the experimental time series $z(t)=Z(Y(t))$ is obtained. Using time delay coordinates with delay $\tau$ and embedding dimension $d$, a $d$-dimensional delay coordinate vector is formed, $\mathbf{X}(t) =(z(t),z(t-\tau),\ldots,z(t-(d-1)\tau)) \in \mathbb{R}^d$. The experimental surface of section is obtained by the common choice that one component of $\mathbf{X}(t)$ equals a constant, e.g., $[\mathbf{X}(t_i)]_1 \equiv z(t_i)=c$. This procedure gives the successive points $\xi_i \in \mathbb{R}^{d-1}$ in the surface of section and the surface of section map $\xi_{i+1}=P(\xi_i)$.

In what follows we focus our interest on the so obtained experimental surface of section map $P$. For the sake of simplicity let us assume that one wants to stabilize an unstable fixed point $\xi_F$ of $P$ which has been localized by the well-known technique of recurrent points [9-11]. Applying the OGY control algorithm implies that one (instantaneously) changes at the times $t_i$ the parameter $p$ from $p_{i-1}$ to an appropriately chosen parameter $p_i$ using (1). Let us now assume that the time between successive piercings of the surface of section is bigger than the lag window, i.e., $t_{i+1}-t_i > (d-1)\tau$. The reason that one hopes to be able to control the original system $Y(t)$ by observing $\mathbf{X}(t)$ is that for appropriately chosen embedding parameters $d$ and $\tau$ [4] there exists a bijective relation $\mathbf{\Phi}$ between the states $\mathbf{X}(t)$ and $Y(t)$, i.e., $\mathbf{X}(t)=\mathbf{\Phi}(Y(t))$. The mapping $\mathbf{\Phi}$ is, however, closely related to the dynamical equations of the system and thus, in general, dependent on the actual value of the control parameter $p_i$. This will be taken into account by writing $\mathbf{\Phi}_{p_i}$ instead of $\mathbf{\Phi}$.

Our argumentation is now as follows. The point $\xi_i$ at time $t_i$ in the surface of section is related to the original state by $Y(t_i)=\mathbf{\Phi}^{-1}_{p_{i-1}}(c,z(t_i-\tau),\ldots,z(t_i-(d-1)\tau))$. Here we make use of our assumption that $(d-1)\tau < t_i - t_{i-1}$ which assures that $p_{i-1}$ is the actual value of $p$ during the whole time interval $(t_{i-1},t_i)$. The time development of the original system from time $t_i$ to the time $t_{i+1}$ is, in case of activated control, given by $\varphi_{p_i}^{t_{i+1}-t_i}$ with $\varphi_p^t$ the flow map of the dynamical system depending on $p$. Thus the state of the system at time $t_{i+1}$ is obtained by $Y(t_{i+1})=\varphi_{p_i}^{t_{i+1}-t_i}(Y(t_i))$ and the corresponding state in the embedding space by $\mathbf{X}(t_{i+1})=\mathbf{\Phi}_{p_i}(Y(t_{i+1}))$. Therefore we obtain $\mathbf{X}(t_{i+1})=(\mathbf{\Phi}_{p_i}\circ\varphi_{p_i}^{t_{i+1}-t_i}\circ\mathbf{\Phi}^{-1}_{p_{i-1}})(\mathbf{X}(t_i))$. This gives our main conclusion. In the case of activated control (i.e., switching the parameter from $p_{i-1}$ to $p_i$ at time $t_i$) the experimental surface of section map $P$ depends not only on the new actual value $p_i$ but also on the preceding value $p_{i-1}$, i.e.,

$$\xi_{i+1}=P(\xi_i,p_{i-1},p_i)\,.$$

Taking this as the starting point the algorithm of OGY is straightforwardly extended. The linearization which one has to consider now is given by

$$\delta\xi_{i+1} \cong A\delta\xi_i + \mathbf{v}\delta p_{i-1} + \mathbf{u}\delta p_i\,,$$

with $A=D_\xi P(\xi_F,p_0,p_0)$, $\mathbf{v}=\partial P/\partial p_{i-1}(\xi_F,p_0,p_0)$, and $\mathbf{u} =\partial P/\partial p_i(\xi_F,p_0,p_0)$. Demanding $\mathbf{f}_u \cdot \delta\xi_{i+1}=0$ one obtains as a new control law

$$\delta p_i = -\frac{\lambda_u}{\mathbf{f}_u \cdot \mathbf{u}}\mathbf{f}_u \cdot \delta\xi_i - \frac{\mathbf{f}_u \cdot \mathbf{v}}{\mathbf{f}_u \cdot \mathbf{u}}\delta p_{i-1}\,. \tag{2}$$

When $P$ is not influenced by the preceding perturbation $\delta p_{i-1}$, i.e., $\mathbf{v}=0$, the original OGY control formula (1) is reobtained. To see this we note that the vector $\mathbf{w}$ in the control formula (1) is related to $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{w}=\mathbf{u}+\mathbf{v}$.

The new control formula (2) contains one possible instability. In the case that $|(\mathbf{f}_u \cdot \mathbf{v})/(\mathbf{f}_u \cdot \mathbf{u})| \geq 1$ holds the required perturbations $\delta p_i$ will, in general, grow until they exceed the maximum allowed value $\delta p_{\max}$, and the range of control will be left. To avoid this instability (i.e., the growing of $\delta p_i$) we propose an alternative approach. We try to find a control law for $\delta p_i$ such that $\delta p_{i+1}$ automatically will become zero. This is done by demanding that the system stabilizes only the next but one step, $i+2$, and that $\delta p_{i+1}$ equals zero, i.e., by the requirements $\mathbf{f}_u \cdot \delta\xi_{i+2}=0$ and $\delta p_{i+1}=0$.

Using the linearization twice, these requirements yield the second modification of the control formula,

$$\delta p_i = -\frac{\lambda_u^2}{\lambda_u \mathbf{f}_u \cdot \mathbf{u} + \mathbf{f}_u \cdot \mathbf{v}}\mathbf{f}_u \cdot \delta\xi_i - \frac{\lambda_u \mathbf{f}_u \cdot \mathbf{v}}{\lambda_u \mathbf{f}_u \cdot \mathbf{u} + \mathbf{f}_u \cdot \mathbf{v}}\delta p_{i-1}\,. \tag{3}$$

The control formulas introduced above have been applied to simulations of a Duffing oscillator [12] given by $\ddot{x}+d\dot{x}+x+x^3=f\cos\omega t$. This system has been numerically integrated. As a measurement function the displacement of the oscillator $z(t)\equiv x(t)$ is chosen. We use a three-dimensional embedding with delay time $\tau=T/4$ with $T=2\pi/\omega$. The experimental surface of section was obtained by taking $[\mathbf{X}(t_i)]_1=z(t_i)=\text{const}$. For the localization of fixed points, the standard method described in Refs. [9,10] is used (for details see also [13]). To obtain the vectors $\mathbf{u}$ and $\mathbf{v}$ the perturbations $\delta p_i$ are alternately switched on and off at every piercing of the surface of section such that $\delta p_i=0$ for $i$ odd and $\delta p_i=\bar{p}$ for $i$ even, $\bar{p}$ small, respectively. Regarding all pairs $(\xi_i,\xi_{i+1})$ with even $i$ as one group and the pairs with odd $i$ as another, it is now possible to fit affine mappings in the neighborhood of $\xi_F$ to $P(\cdot,p_0+\bar{p},p_0)$ using only pairs

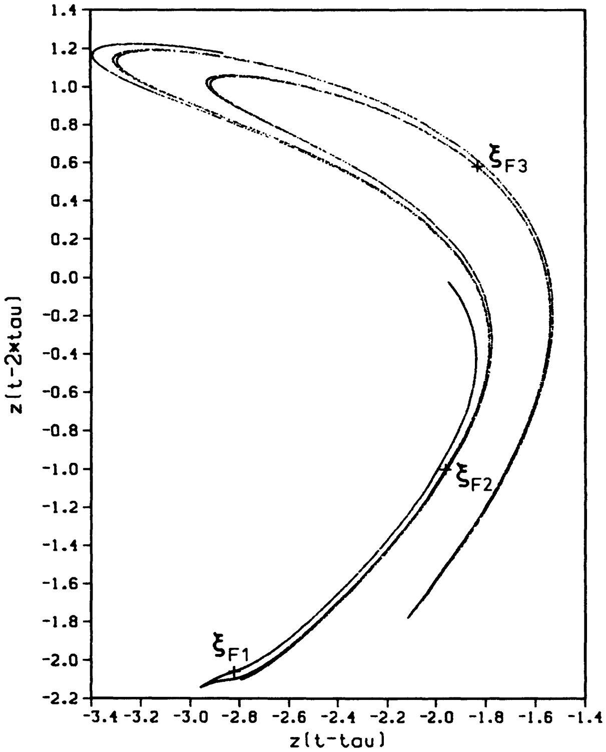


FIG. 1. A chaotic attractor of the Duffing oscillator ($d=0.2$, $f=p=36$, $\omega=0.661$) in the surface of section. The surface of section was obtained by the conditions $z(t_i)=1$, $\dot{z}(t_i)>0$, and $z(t_i-\tau)<0$. The three unstable fixed points observed are indicated by the crosses. For further reference they are called $\xi_{F1}$, $\xi_{F2}$, and $\xi_{F3}$.

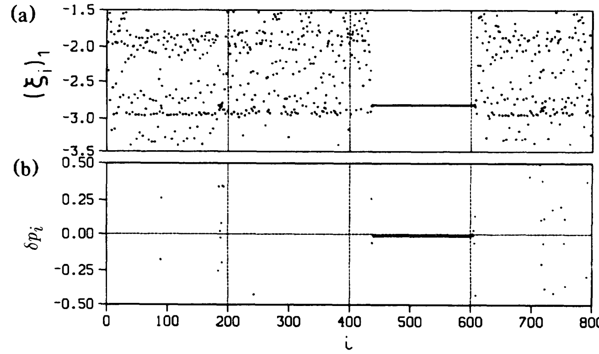


FIG. 2. (a) The first component $(\xi_i)_1$ of the points in the surface of section vs $i$. In order to stabilize the fixed points $\xi_{F1}$ the OGY control formula (1) was switched on from $i=1$ to 200, the first modification (2) from $i=201$ to 400, the second (3) from $i=401$ to 600, and again OGY's control formula from 601 to 800. As can be seen only procedure (3) was able to stabilize $\xi_{F1}$. (b) The parameter perturbations $\delta p_i$ vs $i$ used for control. The maximal allowed disturbance was $\delta p_{\max}=0.5$ and $p_0=36$.

$(\xi_{r_i},\xi_{r_i+1})$, $r_i$ odd, and to $P(\cdot,p_0,p_0+\bar{p})$ using only pairs $(\xi_{r_i},\xi_{r_i+1})$, $r_i$ even, respectively. These fits then determine **u** and **v** by the relations $P(\xi_F,p_0+\bar{p},p_0)\cong\xi_F+\mathbf{v}\bar{p}$ and $P(\xi_F,p_0,p_0+\bar{p})\cong\xi_F+\mathbf{u}\bar{p}$.

To compare the performances of the three different control formulas we tried to stabilize the three fixed points $\xi_{F1}$, $\xi_{F2}$, and $\xi_{F3}$ which were determined embedded in a chaotic attractor of the Duffing oscillator (see Fig. 1). To stabilize these orbits we choose as accessible parameter $p$ the amplitude of the driving $f$ and a maximal allowed perturbation $\delta p_{\max}=0.5$. In Fig. 2, the three different control formulas are successively applied to stabilize $\xi_{F1}$. Only the second modification (3) was able to stabilize $\xi_{F1}$. The coefficients of the control formulas (see Table I) explain why our first modification (2) of the OGY algorithm did not work. The criterion for a stable control algorithm $|b_2|<1$ was hurt. The large absolute value of $b_2=(\mathbf{f}_u\cdot\mathbf{v})/(\mathbf{f}_u\cdot\mathbf{u})$, indicates further that the influence of the change of the preceding parameter $p_{i-1}$ is relatively larger than that of the actual one $p_i$. But this is exactly what is neglected if one applies the original approach of OGY without considering the meaning of the time delay coordinates.

The stabilization of the second fixed point shows different features. Here the generic condition $(\mathbf{f}_u\cdot\mathbf{w}\neq0)$ of the OGY formula is almost violated. Because of the resulting large value of the coefficient $a$ there were only rare cases where the control requirement $\delta p_i<\delta p_{\max}$ was met. But even then the control range was soon left without succeeding in control. The coefficient $b_2$ just violates the stability criterion. Indeed, the used perturbations $\delta p_i$ increased at the beginning. But finally, probably due to nonlinear effects, the control procedure stabilized and the algorithm was able to achieve control. The second modification (3) was again able to achieve control but with perturbations drastically smaller than the one used for (2).

The third fixed point $\xi_{F3}$ could be stabilized by any of the three versions of the control formula. For $\xi_{F3}$ the coefficients of the control formulas are very similar (see Table I). The coefficient $b_2$ is relatively small which indicates the small influence of $\delta p_{i-1}$ compared to $\delta p_i$. So one can expect that all three algorithms will work.

The algorithms were also tested using further surfaces of section. Among others we also investigated the stroboscopic surface of section map which was used by Ditto, Rauseo, and Spano in [5], i.e., as time series we took

TABLE I. The numerically obtained values of the coefficients in the control formulas for the three fixed points considered. The coefficients are introduced implicitly by writing the OGY control formula as $\delta p_i=a\hat{\mathbf{f}}_u\cdot\delta\xi_i$, the first modification (2) as $\delta p_i=b_1\hat{\mathbf{f}}_u\cdot\delta\xi_i+b_2\delta p_{i-1}$, and the second modification (3) as $\delta p_i=c_1\hat{\mathbf{f}}_u\cdot\delta\xi_i+c_2\delta p_{i-1}$ with $\hat{\mathbf{f}}_u=\mathbf{f}_u/\|\mathbf{f}_u\|$.

| | $a$ | $b_1$ | $b_2$ | $c_1$ | $c_2$ | $\lambda_u$ |
|---|---|---|---|---|---|---|
| $\xi_{F1}$ | −16.43 | −228.4 | −12.9 | 38.9 | 2.2 | −1.87 |
| $\xi_{F2}$ | 164.6 | −7.35 | 1.04 | −9.4 | 1.33 | 4.82 |
| $\xi_{F3}$ | −2.46 | −1.97 | 0.20 | −1.79 | 0.18 | −1.85 |

a stroboscopic measurement $x(t_i)$, $t_i - t_{i-1} = T$, and obtained a surface of section with points $\xi_i = (x(t_i), x(t_{i-1}))$. In this surface of section the periodic motion corresponding to $\xi_{F1}$ could be stabilized by all three algorithms. They were almost equivalent because $b_2$ and $c_2$ were nearly zero (of the order of $10^{-4}$), so the other coefficients were practically the same ($a \approx b_1 \approx c_1 \approx 2.7$). The periodic motion corresponding to $\xi_{F2}$ could not be stabilized because the embedding in the neighborhood of the fixed point was bad (not injective). The third fixed point finally could only be stabilized using the second modification (3).

Altogether the numerical investigations show that the possibility of stabilizing a fixed point is not an intrinsic property of a fixed point, as the eigenvalues $\lambda_u$ and $\lambda_s$ are, for example. The coefficients of the control formulas differ for different surfaces of section and so do their performances. We always observed that the performance of the first modification (2) is superior to the one of the original OGY formula and the second modification outperforms the latter two. However, their performances are similar whenever the influence of the preceding parameter perturbation $\delta p_{i-1}$ is small which results in a small value of $\mathbf{f}_u \cdot \mathbf{v}$. But we did observe that the OGY formula failed and the applications of one of the modifications could stabilize the desired fixed point. As a rule this happened when the influence of the changes of the preceding parameter was noticeable, which resulted in a non-negligible value of $\mathbf{f}_u \cdot \mathbf{v}$.

In conclusion, we introduced two modifications of the control formula of OGY which can lead to a better performance of the control in the case that the dynamical system is reconstructed using time delay coordinates. Therefore these modifications extend the range of applicability of the OGY control method. With these modifications all remarkable advantages of the OGY control method are preserved; e.g., the dynamics equation is not required, the perturbations of the accessible parameter can be very small, different periodic points can be stabilized in the same parameter range for the same system, and after having determined the control coefficients the computational effort at every iteration is negligible which opens the possibility of real time applications. We expect that the OGY control method will yield important applications in the future for technical systems also.

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[8] We note that the OGY control formula looks different from the one in Ref. [1]. The reason for this is that we use the linearization around $\xi_F$ and $p_0$ (as is used in [6]) and do not estimate the new position of the fixed point $\xi_F(p)$ as one changes the parameter $p$, $\xi_F(p) = \xi_F(p_0) + \mathbf{g}\delta p$, as OGY do. But it is easy to show that these approaches are equivalent and $\mathbf{g}$ and $\mathbf{w}$ are related through $\mathbf{g} = [1 - D_\xi P(\xi F, p_0)]^{-1}\mathbf{w}$.

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