Pseudoperiodic Driving: Eliminating Multiple Domains of Attraction Using Chaos

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The basins of attraction of period-2 attractors can be subdivided into two domains, one for each of the stable trajectories which are one drive cycle apart. This generalizes to period-n attractors. Periodic drive signals can be replaced by certain chaotic signals which result in the elimination of multiple domains of attraction. The attractors are similar to the original ones, but two systems on the same attractor cannot get out of phase with each other.

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When nonlinear systems are driven with simple periodic signals even nonchaotic behavior can be complex, such as period doubling, tripling, etc. [1]. This can lead to difficulties in obtaining system behavior in which driven components are in phase. The basin of attraction of a period-n trajectory can be subdivided into n *domains,* one for each of the final, out-of-phase trajectories on the attractor. Thus, several nonlinear systems driven with the same periodic signal can be in a stable situation in which they are out of phase with each other. This problem can be compounded in a system with a fractal basin boundary. Here prediction of the final system state (phase) can be very difficult, as McDonald, Grebogi, Ott, and Yorke [2] have pointed out, since the fractal structure gives an uncertainty to determining the domain of the initial conditions which is difficult to eliminate. We propose a way to use chaos to eliminate these problems.

Very little work has been done on driving nonlinear systems with signals that are chaotic. Some exceptions are recent work by Hübler and co-workers [3,4] and by Pecora and Carroll [5–8]. We address in this paper, for the first time, the questions of basins of attractions, stability, and dynamical behavior of a large set of such systems.

In particular, we show that it is possible to add certain chaotic signals to the periodic drive to eliminate multiple domains, but retain the stability and the general dynamical topology of the trajectory. The overall shape of basins of attraction will be retained, but will be simplified. This guarantees that devices or systems driven with these signals will always synchronize (be in phase) and prediction of their final state will be more accurate, yet their behavior will be almost the same as in the periodically driven case.

Theory.— In the following we give a heuristic argument for the simplification of period-n-attractor basins. We refer to the origin of the drive signal as the *drive* system and the driven system as the *response* system. We define a *basin of attraction* as the set of all points in phase space that converge to a particular attractor and a *domain of attraction* as the set of points converging asymptotically to a particular final trajectory. For example, a period-2 attractor will have its basin divided into two domains.

We consider a driven dynamical system $\dot{w}=f(w,v)$, where w and f are n-dimensional vectors and functions and v is a periodic driving signal. Then we change v *slightly* to a new driving signal v'. If $w'(t_0)$ is an initial point in the v'-driven system nearby $w(t_0)$, their difference will evolve according to

$$\frac{d(w'-w)}{dt}\equiv\Delta\dot{w}=f(w',v')-f(w,v)\,, \qquad (1)$$

which, by adding and subtracting $f(w,v')$, can be rewritten as

$$\Delta\dot{w}=D_wf(w,v)\Delta w+B(t)\,, \qquad (2)$$

where D_wf is the Jacobian of the vector field, $B(t)=f(w,v')-f(w,v)$, and we have dropped the higher-order terms for now. Equation (2) is a linear equation. Its solution can be given in terms of the transfer function [9] $\Phi(t,t_0)$ for the homogeneous version of Eq. (2), viz.,

$$\Delta w(t)=\Phi(t,t_0)\Delta w(t_0)+\int_{t_0}^{t}\Phi(t,\tau)B(\tau)d\tau\,. \qquad (3)$$

If the original response system is stable, it will have negative conditional Lyapunov exponents [5–7,10]. This means [9] there exist two constants $c_1>0$ and $c_2>0$ such that $\|\Phi\|\le c_1e^{-c_2t}$. If $B(t)$ is bounded by a constant $b_1>0$ (which it will be for many cases), $|\Delta w(t)|\le c_1b_1/c_2$ for large t.

For small deviations from the original periodic drive (b_1 small) the trajectory will always remain close to the original trajectory—multiple-period attractors will remain nearly so. For larger deviations from the original drive the above analysis will not be valid for long times since we cannot drop higher-order terms and b_1 may be large enough that Δx can become on the order of the attractor size. So we should expect a threshold above which the behavior of the system will cease to be always close to that of the original response.

However, the new behavior above the threshold may still resemble the original response if the new response motion remains stable with respect to the new drive v' and the new drive is still not too different from the original. Hence, we will use a chaotic drive with spectral features similar to the original periodic drive. But, since v' has a chaotic component, the periodicity will be lost and multiple-period behavior and multiple domains will not be possible. We use the term *pseudoperiodic drive* to describe this drive signal.

Certain chaotic systems (e.g., the Rössler) can have very sharp spectral peaks. By tuning them so that their peaks match the frequencies of the original periodic drive we have candidates for just the drives we need. The addition of chaos can be accomplished by simply replacing the periodic drive with the output from such a chaotic system [8] or in a more controlled fashion by adding the chaos to the periodic drive, viz.,

$$v'(t) = v(t) + \varepsilon x(t)\,, \tag{4}$$

with ε variable and $x(t)$ a dynamical variable from the chaotic system, which also serves to demonstrate the threshold phenomena.

Numerical experiments.— We chose the Duffing system [1] (Ueda version) as the response. The pseudoperiodic drive was a cosine plus the x component from a Rössler system which was tuned to have its large spectral peak at the same frequency as the cosine. The equations of motion are

$$\frac{dw_1}{dt} = w_2,\quad \frac{dw_2}{dt} = -kw_2 + w_1^3 + \alpha v' + \beta\,, \tag{5}$$

where v' is as in Eq. (4). We use $v(t) = \cos(t)$, $k = 0.05$, $\alpha = 0.21$, $\beta = 0.15$, $a = b = 0.2$, and $c = 4.5$. For these parameters the cosine-driven Duffing system has period-1, period-2, and period-3 attractors coexisting, with period-2 and period-3 overlapping in the 2D response w subspace [11].

Let us examine one case first, $\varepsilon = 0.129$. The period-3 attractor ceases to exist (becomes unstable), which we comment on later. The period-1 remains and the period-2 loses its multiplicity— it changes to an attractor which appears much like the original period-2, but which has only one domain of attraction. We call this a pseudoperiod-2 attractor. Figure 1 shows the attractors of the cosine-driven and the pseudoperiodically driven Duffing system along with the period-2 and pseudoperiod-2 time series. The latter go for ~ 30 cosine cycles before getting out of phase (as they must), but the pseudoperiodic trajectory continues to mimic the period-2 behavior forever. Hence, we can get the Duffing system to behave much like a period-2 attractor, but without the multiplicity.

Figure 2(a) shows the domains of attraction for the period-1, -2, and -3 attractors for the cosine-driven Duffing system. The situation is rather complicated, with six different domains of attraction. There is also evidence that there are fractal basin boundaries, although we have not studied this in detail.

Figure 2(b) shows the basins of attraction for the pseudoperiod-1 and -2 attractors [1]. The situation is greatly simplified. All initial conditions for the period-3 attractor have been converted to pseudoperiod-2 basin points. Only one pseudoperiod-2 domain exists. However, the overall shape of the pseudoperiod-2 basin is very close to the combined basins of the period-2 and -3 attractors in Fig. 2(a) without the apparent fractal structure.

We have explored this system for other ε values and for other Rössler c parameters which change the spectral nature of the chaos. In general we find that there is a threshold (ε value) above which the period-3 orbit becomes unstable (going to a period-2) and, simultaneously, the period-2 loses its multiplicity. The number of cosine cycles (averaged over the pseudoperiod-2 basin) for the Duffing system to converge to the pseudoperiod-2 trajectory as a function of ε scaled as $1/(\varepsilon - \varepsilon_c)^{\nu}$ above threshold, with $\varepsilon_c = 0.0154$ and $\nu = 0.955$. The ε_c threshold amounts to adding only a few percent chaos (in terms of amplitudes) to the cosine drive to eliminate multiplicity. This scaling is like that for transient chaos [8,12,13]. This suggests that the loss of attractor multiplicity comes

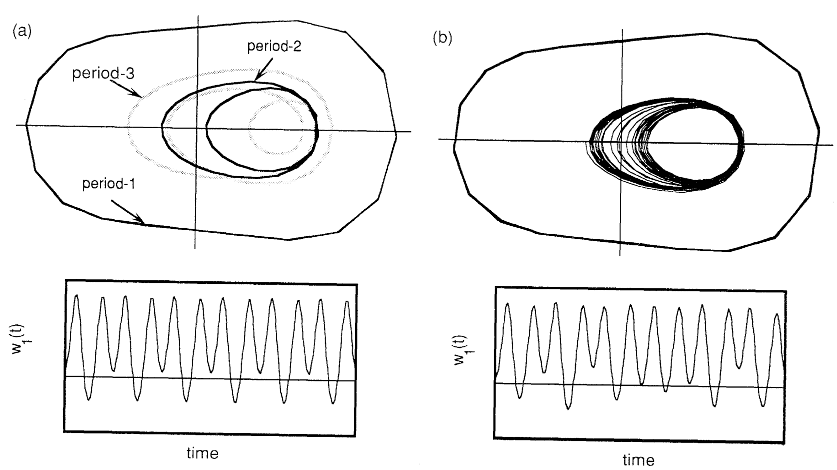

FIG. 1. Attractors and period-2 time series for (a) periodically driven and (b) pseudoperiodically driven Duffing system.

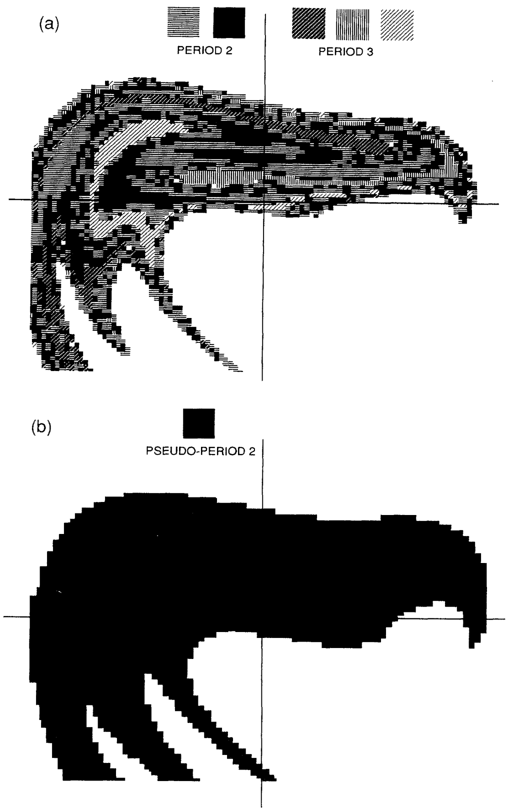

FIG. 2. (a) Basins (domains) of attraction for cosine-driven Duffing system. (Everything outside of the hatched area goes to period-1.) (b) Basins of attraction for pseudoperiodic driven Duffing system.

from a crisis.

For Rössler c values in which the chaos is more broadband (above $c \approx 8.3$) we also find thresholds in the same place, but the Duffing system often goes unstable (a positive conditional Lyapunov exponent [5–7]) at moderate ε values and the nature of the trajectories does not emulate the period-2 attractor as well. Hence, spectral similarity and, perhaps, small positive Lyapunov exponent for the chaotic component of the drive appear necessary for emulation of the multiple-period behavior and the stability of pseudoperiodic trajectories.

The loss of stability of the period-3 Duffing at the threshold prompts us to conjecture that m any two-dimensional system overlapping trajectories cannot coexist above threshold. The response system is no longer periodic, meaning that the system's phase-space points on each trajectory can come arbitrarily close to each other at their crossing. If both are assumed stable this would lead to a contradiction in having nearby points asymptotically stable to two different attractors. We are examining this in more detail presently [8]. In higher dimensions attractor crossing is not generic and destruction of attractors will probably not happen this way.

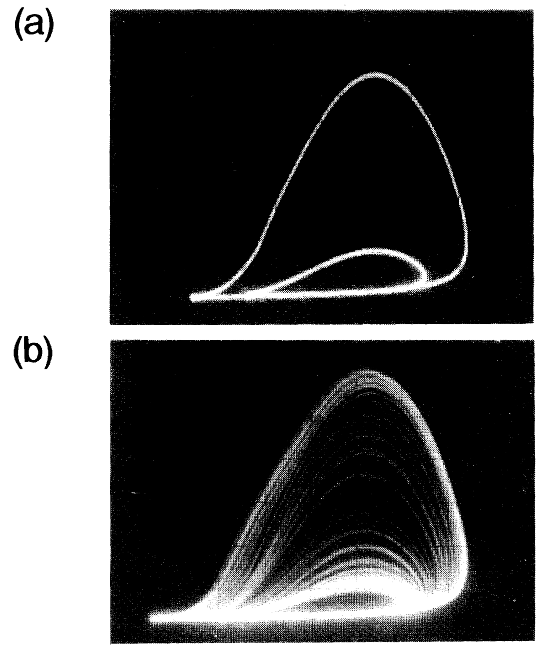

FIG. 3. Oscilloscope pictures of (a) period-2 and (b) pseudoperiod-2 circuit responses.

Electrical circuit demonstration.— To test multiple-domain elimination using pseudoperiodic driving on a real system, we built two closely matched nonautonomous nonlinear circuits. The differential equations describing these circuits were similar to those describing a Rössler attractor, except that we removed the y component of these equations and replaced it with an external sinusoidal drive. The driving frequency was 2.02 kHz. The equations for this system are

$$\alpha\dot{x} = A\cos\omega t + \text{const} - z, \quad \alpha\dot{z} = b + xz - cz\,, \qquad (6)$$

where $\alpha = 10^4$, $b = 0.25$, and $c = 2.94$.

We first drove a pair of these circuits with the sum of the 2.02-kHz cosine wave and the x component of an autonomous 3D Rössler circuit with a spectral peak near 2.02 kHz. We set the two response circuits initially to be period doubled ($A = 0.495$ V and const $= 15$ mV) and set the initial conditions to make the two systems out of phase. We observed the phase relation between the two response systems by plotting the x component of one versus the x component of the other on an oscilloscope.

As we began to add the chaotic Rössler signal to the cosine drive, we observed a slight broadening in the attractor for the period-doubled response systems. When the amplitude of the added chaotic Rössler signal exceeded a threshold of about 9% of the amplitude of the cosine drive, we observed the two response circuits become in phase. Above this threshold, we were not able to reset the two systems to remain out of phase. Figure 3 is an oscilloscope photograph of the attractors for one of the response systems. The attractor for this system still resembles the original period-doubled attractor.

Since it is impossible to make two perfectly identical circuits, the signals from the two response circuits were never identical. We used an indirect method to measure how closely the signals from these two circuits came to being in phase. We sampled the x component from each response circuit at the point at which the cosine component of the drive crossed zero in the positive direction. These Poincaré sections for cosine driving consisted of two points. We drew a perpendicular bisector to the line between these two points to divide the Poincaré sections into two regions. We then counted the fraction of times that the two response systems were in corresponding regions during pseudoperiodic driving. This could only happen when the two systems were in phase.

We found that when the chaotic Rössler signal reaches about 9% of the drive, the two systems become in-phase about half of the time. We believe that the reason that the in-phase fraction does not rise quickly to 100% is that when driving in this regime, the response systems are close to being unstable (chaotic). Small amounts of noise may cause the two systems to momentarily diverge from each other, throwing them out of phase until they are able to approach each other again. With larger amounts of chaotic component (20%–30%) the percentage of time the responses remained in phase rose to 80%–90% [8].

We also found that many other chaotic circuits were good candidates for use in pseudoperiodic drives and gave similar results. These will be reported on elsewhere [8]. We also attempted to simplify the domain structure by driving with quasiperiodic signals. The latter was sensitive to drive parameter changes (a slight shift in frequency changes a quasiperiodic to a periodic signal). This resulted in beating behavior of the response attractor and often retention of multiple-period behavior.

Conclusions.— There are several striking features of pseudoperiodic driving above threshold. One is robustness. The stable region spans at least an order of magnitude in ε and is not sensitive to changes in the driving system, provided the drive remains chaotic. Another feature is that pseudoperiodic driving results in smooth response behavior which closely mimics (forever) the response behavior with a periodic drive. Finally, we often need add only a few percent of the chaotic signal to a periodic signal to eliminate multiple-period behavior.

Multiple-period behavior is ubiquitous in nonlinear systems. This provides a broad range of potential applications wherever out-of-phase behavior is deleterious. These include driven modes in spatial-temporal systems, driven systems of coupled oscillators [14–18], neural nets with periodic input, nonlinear electrical circuits needing nearly periodic synchronization pulses, and biological systems [19–21]. If one wants to drive a complex nonlinear system so that all subsystems are, in some way, stably synchronized, then the driving signal of choice would not be a periodic one, but rather a pseudoperiodic one.

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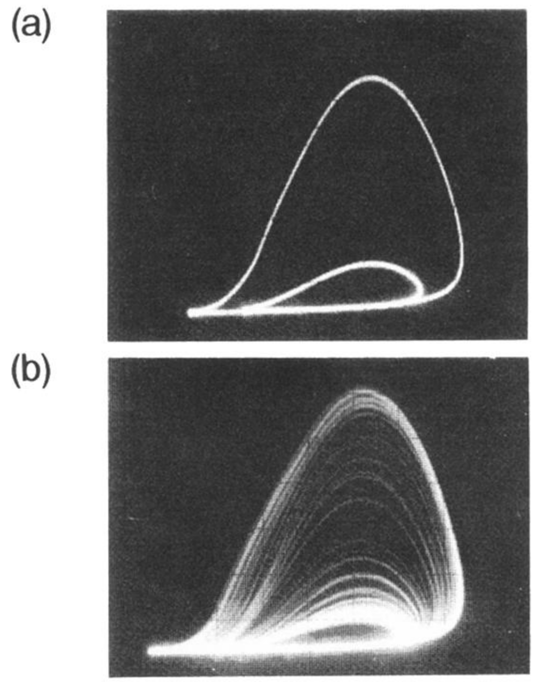
[1] This type of behavior was well known long before chaotic behavior. See A. A. Andronow and S. E. Chaikin, in *Theory of Oscillations* (Princeton Univ. Press, Princeton, NJ, 1949), for early examples, and later C. Hayashi, in *Nonlinear Oscillations in Physical Systems* (Princeton Univ. Press, Princeton, NJ, 1964), and references therein.
[2] S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica (Amsterdam) **17D**, 125 (1985).
[3] A. Hübler and E. Lüscher, Naturwissenschaften **76**, 67 (1989).
[4] G. Reiser, A. Hübler, and E. Lüscher, Z. Naturforsch. **42a**, 803 (1987).
[5] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[6] L. M. Pecora and T. L. Carroll, IEEE Trans. CAS **38**, 453 (1991).
[7] L. M. Pecora and T. L. Carroll, Phys. Rev. A (to be published).
[8] L. M. Pecora and T. L. Carroll (to be published).
[9] T. A. Burton, in *Stability and Periodic Solutions of Ordinary and Functional Differential Equations* (Academic, New York, 1985).
[10] J.-P. Eckmann and D. Ruelle, Rev. Mod. Phys. **57**, 617 (1985).
[11] The systems were integrated using a Runge-Kutta 4-5 method. The Lyapunov exponents were calculated according to Ref. [10]. The domains of attraction for the pseudoperiodically driven systems were calculated by integrating two initial conditions forward in time. One was started at a point known to converge to a particular attractor; the other was started at an arbitrary point in phase space. Convergence of the latter to the particular attractor was accepted when both trajectories remained within 2%-5% of each other for more than 60 cosine cycles. Tests for convergence with over 5000 cycles were also done to insure stability.
[12] C. Grebogi, E. Ott, and J. A. Yorke, Ergodic Theory Dynamical Systems **5**, 341 (1985); Phys. Rev. Lett **57**, 1284 (1986).
[13] T. L. Carroll, L. M. Pecora, and F. J. Rachford, Phys. Rev. Lett. **59**, 2891 (1987).
[14] A. Atiya and P. Baldi, Int. J. Neural Sys. (to be published).
[15] E. E. Shnol, Prikl. Mat. Mekh. **51**, 15–20 (1987).
[16] H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, Prog. Theor. Phys. **77**, 5 (1987).
[17] N. Kopell and G. B. Ermentrout, Math. Biosciences **90**, 87 (1988).
[18] R. Loft and T. A. DeGrand, Phys. Rev. B **35**, 8528 (1987).
[19] A. Garfinkel, Am. J. Physiol. **245**, R455 (1983).
[20] L. Glass, A. Goldberger, M. Courtemanche, and A. Shrier, Proc. Roy. Soc. London A **413**, 9 (1987).
[21] C. Skarda and W. J. Freeman, Behav. Brain Sci. **10**, 161 (1987), and the commentaries following the article.



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