Eigenvalue Statistics of Random Real Matrices

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Completing Ginibre's work we determine the joint probability density of eigenvalues in a Gaussian ensemble of real asymmetric matrices, which is invariant under orthogonal transformations. The symmetry parameter τ may vary from -1 (antisymmetric ensemble) through 0 (completely asymmetric ensemble) to $+1$ (symmetric ensemble). The elliptic law for the average density of eigenvalues in the limit of large dimension is recovered. Matrices of the type considered appear in models for neural-network dynamics and dissipative quantum dynamics.

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Since the early 1950s, random matrix theory has developed into a powerful tool in statistical mechanics [1]. Besides the classical examples—statistical properties of nuclear and atomic spectra—now stand applications concerning quantum signatures of chaos [2]. Random matrix results are also encountered in the theory of spin glasses and neural nets.

In the Sherrington-Kirkpatrick model of spin glasses [3] symmetric exchange interactions J_{ij} $(=J_{ji})$ between N Ising spins are random variables with zero mean and variance $\langle J_{ij}^2\rangle = 1/N$, $i\neq j$. In the large-N limit the average eigenvalue density is Wigner's semicircle law [1] with a radius of 2 on the real axis. For neural networks Ising spin-flip dynamics has been applied to neuron dynamics [4] with (potentially asymmetric) synaptic efficacies J_{ij}, and for a simple model the eigenvalue density has been studied for covariances $\langle J_{ij}^2\rangle = 1/N$ and $\langle J_{ij}J_{ji}\rangle = \tau/N$ ($i\neq j$ and $-1\leq\tau\leq+1$). One finds [5] for large N a homogeneous eigenvalue distribution in the complex plane, in an elliptic shape, centered around the origin with half axes $1+\tau$ and $1-\tau$. The eigenvectors play an important role in network dynamics especially if analog neurons with variable nonlinearity are considered [6].

Random asymmetric real matrices also occur as generators of dissipative quantum dynamics. It has been found [7] that statistical properties of the eigenvalues for large N are well described by the Gaussian ensemble of general complex (non-Hermitian) matrices investigated by Ginibre [8]. It is one of the goals of our paper to justify this conjecture. In order to determine the degree of eigenvalue repulsion in the complex plane for real asymmetric matrices, an almost-degenerate perturbation-theory argument was given in [9], together with a plausible assumption on the probability density of the occurring matrix elements. Now, the degree of level repulsion can be given rigorously.

One of the most important successes of random matrix theory is, for given matrix ensembles, the determination of the joint probability density of all eigenvalues. From this, all correlation functions (including the average eigenvalue density) can be derived. Especially, for small eigenvalue differences the correlation functions show the above-mentioned property of level repulsion present in quantum systems which behave classically chaotic. Requirements of statistical independence for unrestricted matrix elements and of invariance with respect to certain symmetry groups lead to Gaussian matrix ensembles. Three different such ensembles of Hermitian matrices have to be studied, when nonintegrable conservative quantum systems are dealt with [10]. In these cases the joint probability density of (real) eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_N$, which is obtained by integrating out all eigenvectors, is of the form

$$P(\lambda_1,\lambda_2,\ldots,\lambda_N) = A\prod_{i<j}|\lambda_i-\lambda_j|^\beta \exp\left(-B\sum_i \lambda_i^2\right). \quad (1)$$

Here the constants A and B set normalization and scale, and the exponent β, which determines the degree of level repulsion, takes the values 1, 2, and 4 for real, complex, and quaternion real Hermitian matrices, respectively. The corresponding Gaussian ensembles are called orthogonal (GOE), unitary (GUE), and symplectic (GSE) because their members are invariant under the indicated transformations. From the mathematical point of view it was a natural task undertaken by Ginibre [8] to calculate the joint probability density of eigenvalues in the matrix ensembles that arise if the Hermiticity condition is dropped in GOE, GUE, and GSE. For general complex and general quaternion real matrices with independently Gaussian distributed matrix elements, the problem was solved. The result for complex matrices is again of the form (with certain constants A and B)

$$P(\lambda_1,\lambda_2,\ldots,\lambda_N) = A\prod_{i<j}|\lambda_i-\lambda_j|^2 \exp\left(-B\sum_i |\lambda_i|^2\right). \quad (2)$$

From (2) one derives the nearest-neighbor-distance distribution [7], which goes as $\propto S^3$ for small distances S. The extra factor S in the level repulsion, as compared to (2), is contributed by the two-dimensional volume element in the complex plane.

For asymmetric real matrices the restricted case of real eigenvalues which could be treated completely by Ginibre yields (1) with $\beta=1$. In accordance with the conjecture in Ref. [7] (see above), we should expect in the general case a joint probability density which shows certain aspects of both distribution (1) and distribution (2). In the following we will fill the gap and derive the desired general joint probability density for real asymmetric matrices. For simplicity, we will treat the case of independent matrix elements with $\langle J_{ij}^2\rangle = 1$ explicitly and later we will generalize to correlated matrix elements and more general covariances using scale transformations.

We start with the normalized Gaussian measure

$$d\mu(J) = (2\pi)^{-N^2/2} \exp\left(-\sum_{ij} J_{ij}^2/2\right) \prod_{ij} dJ_{ij}\,. \tag{3}$$

We try to find the joint probability density of eigenvalues of the matrix J_{ij}, $i,j \in \{1,2,\ldots,N\}$, with the only restriction being that all J_{ij} are real. For an eigenvalue λ of the matrix J_{ij} there exists an eigenvector $\xi_j \neq 0$, such that

$$\sum_j J_{ij}\xi_j = \lambda \xi_j\,. \tag{4}$$

Since J_{ij} is real λ^* is also an eigenvalue with eigenvector ξ_j^*. Hence the eigenvalues are real or complex conjugate in pairs. Following from (4), λ is a zero of the characteristic polynomial

$$\det(J_{ij} - \lambda\delta_{ij}) = 0 = \prod_{i=1}^{N} (\lambda_i - \lambda)\,. \tag{5}$$

In the generalization of (4) we may define the eigenvalues as the N zeros of the polynomial (5). We do not consider exceptional cases where eigenvalues cross, because these have measure zero [8]. We especially exclude cases where eigenvectors collapse, so that we have R real eigenvalues and Q complex-conjugate pairs with $N = R + 2Q$. From elementary linear algebra we know [8] that a real matrix J with distinct eigenvalues can be transformed by a nonsingular real matrix X,

$$J = X\tilde{\Lambda}X^{-1}, \quad \det(X) \neq 0\,, \tag{6}$$

so that $\tilde{\Lambda}$ takes the form

$$\tilde{\Lambda} = U\Lambda U^{-1}, \quad \Lambda_{ij} = \lambda_i \delta_{ij}\,, \tag{7}$$

and the unitary matrix U is

$$U = \begin{pmatrix} 1 & & 0 & \vdots & & & \\ & \ddots & & \vdots & & 0 & \\ 0 & & 1 & \vdots & & & \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & & \vdots & a\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} & & 0 \\ & 0 & & \vdots & & \ddots & \\ & & & \vdots & 0 & & a\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{pmatrix}, \tag{8}$$

with $a = 1/\sqrt{2}$. Here we assume that the first R eigenvalues are real and the next Q pairs are complex conjugate. The two-dimensional blocks in (8) transform the complex-conjugate eigenvalues $\lambda = a + ib$, $\lambda^* = a - ib$ into a real matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{9}$$

So $\tilde{\Lambda}$ consists of a diagonal part due to the real eigenvalues and Q blocks of type (9) due to the complex-conjugate pairs. As one sees easily, the columns of the matrix XU are the eigenvectors of the matrix J. Hence they can be chosen real for real eigenvalues and complex conjugate for complex-conjugate pairs of eigenvalues.

From linear algebra we know as well that the transformation matrix X can be written as $X = OY\tilde{D}$. Here $O \in \mathrm{O}(N)$ is a real orthogonal matrix, $Y \in \mathrm{Y}(N)$ is a real upper triangular matrix with diagonal elements 1, and $\tilde{D} \in \tilde{\mathrm{D}}(R,Q)$ is a matrix of the form of $\tilde{\Lambda}$ with $\det(\tilde{D}) \neq 0$. Originally one has the relation $X = OYD$, where D is diagonal [8]; however, this is readily generalized to the given equation.

$\mathrm{O}(N)$, $\mathrm{Y}(N)$, and $\tilde{\mathrm{D}}(R,Q)$ are groups with respect to matrix multiplication. It is rather obvious that the 2×2 matrices of type (9) are isomorphic to the field of complex numbers; therefore all matrices $\tilde{D}$ commute with $\tilde{\Lambda}$. Hence $\tilde{D}$ disappears from the representation (6) of J. Now we have

$$J = O\tilde{J}O^{-1} \text{ with } \tilde{J} = Y\tilde{\Lambda}Y^{-1}\,. \tag{10}$$

The corresponding eigenvectors can be chosen as the columns of the matrix OYU. Counting independent parameters we find N^2 for J, N for $\tilde{\Lambda}$, and $N(N-1)/2$ for both $O \in \mathrm{O}(N)$ and $Y \in \mathrm{Y}(N)$. Thus we may ask whether representation (10) is unique for a special order of eigenvalues.

Let us suppose in the following without loss of generality that the eigenvalues in (7) are ordered according to

$$\lambda_1 < \lambda_2 < \cdots < \lambda_R, \quad a_1 < a_2 < \cdots < a_Q, \quad b_j > 0\,. \tag{11}$$

Here and below considerations can mostly be restricted to the two-dimensional blocks along the diagonal, where Y takes the form

$$Y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}. \tag{12}$$

We denote the off-diagonal element by y. It can be shown that by two rotations allowed by (10) (multiplying Y from left and right) y can be transformed to $-y$. Choosing y positive in the following, the only nonuniqueness in a representation (10) of J that obeys (11) is due to 2^{R+Q} inflections S ($S^2 = 1$, including unity) that commute with $\tilde{\Lambda}$: replacing O and Y by $O' = OS$ and $Y' = SYS$, respectively, we get the same matrix J for given $\tilde{\Lambda}$.

We now transform the Gaussian measure (3). The exponent

$$\sum_{ij} J_{ij}^2 = \mathrm{Tr}(JJ^T) = \sum_{ij} \tilde{J}_{ij}^2 \tag{13}$$

is invariant under orthogonal transformations; therefore integration over the orthogonal group O(N) should be simple. In a two-dimensional block along the diagonal, $\tilde{J}$ is

$$\tilde{J} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a-yb & b(1+y^2) \\ -b & a+yb \end{pmatrix} . \tag{14}$$

All further elements $\tilde{J}_{ij}$ below the diagonal (e.g., $i>j$) vanish. Setting $j=b(1+y^2)>b$, this gives a contribution

$$2(a^2-b^2)+(j+b)^2 \tag{15}$$

to the sum (13). Therefore it is most convenient to consider Λ, O, and $\tilde{J}_{ij}$ (for $i<j$) as the independent variables. If we manage to integrate out O and $\tilde{J}_{ij}$ (for $i<j$), we will have the desired joint probability density of eigenvalues.

It is indeed possible to change the independent variables and transform the Lebesgue measure in (3) using the representation (10). Let us introduce

$$d^{(R,Q)}\lambda = \prod_{i<j}^{1\cdots R} d\lambda_i \prod_{k=1}^{Q} 2\,da_k\,db_k \,. \tag{16}$$

A procedure similar to the one used by Ginibre [8] leads to

$$\prod_{ij} dJ_{ij} = d\Omega(N) \prod_{i<j} d\tilde{J}_{ij} \prod_{i<j} |\lambda_i-\lambda_j| d^{(R,Q)}\lambda \,, \tag{17}$$

where $d\Omega(N)$ is the invariant measure [10] of the group O(N),

$$\Omega(N) = \int d\Omega(N) = \prod_{d=1}^{N} \frac{2\pi^{d/2}}{\Gamma(d/2)} \,. \tag{18}$$

To achieve (17) we first calculate the increment dJ using the representation (10),

$$dJ = \mathcal{S}\{[U^{-1}Y^{-1}(O^{-1}dO + dY\,Y^{-1})YU, \Lambda] + d\Lambda\}\mathcal{S}^{-1} \,, \tag{19}$$

with $\mathcal{S}=OYU$. The Lebesgue measure (17) is invariant under a similarity transformation $\mathcal{S}$ of J; therefore we may immediately separate $d^{(R,Q)}\lambda$ due to the diagonal part $d\Lambda$. In an intermediate step we find

$$\prod_{ij} dJ_{ij} = d\Omega(N) \prod_{k=1}^{Q} y_k \prod_{i<j} dY_{ij} \prod_{i\neq j} |\lambda_i-\lambda_j|^2 d^{(R,Q)}\lambda \tag{20}$$

and for fixed λ_i,

$$\prod_{i<j} d\tilde{J}_{ij} = \prod_{k=1}^{Q} y_k \prod_{i<j} dY_{ij} \prod_{i<j} |\lambda_i-\lambda_j| \,. \tag{21}$$

The calculations may conveniently be done with the help of the calculus of alternating differential forms [11]. Inserting (21) into (20) we obtain the important equation (17). The appearance of the off-diagonal elements y_k in the diagonal blocks (12) of Y may be understood from the two-dimensional case. Then $O^{-1}dO + dY\,Y^{-1} + Y d\tilde{\Lambda} Y^{-1}$ which appears to be relevant for (17) has the form

$$\begin{pmatrix} da - y\,db & (1+y^2)db + dy + d\phi \\ -d\phi - db & da + y\,db \end{pmatrix} \tag{22}$$

and therefore

$$\prod_{ij} dJ_{ij} = d\phi\, y\, dy\, 2\,da\,db |\lambda-\lambda^*|^2 \,. \tag{23}$$

Here $d\phi$ is the off-diagonal element of the antisymmetric matrix $O^{-1}dO$.

With the help of the transformed Lebesgue measure (17) the integration over the eigenvectors, i.e., dO and dY, is simple. Since there is uniqueness only up to inflections and we want to integrate over the whole group O(N), we have to divide by 2^{R+Q}. The integration over $\tilde{J}_{ij}$ outside the two-dimensional blocks (14) is simply Gaussian. There remains the integration for $\tilde{J}_{ij}$ inside a block. Using (15) we get for each block a term

$$\int_b^\infty dj \exp[-(a^2-b^2)-(j+b)^2/2] = \sqrt{\pi/2}\exp[-(a^2-b^2)]\mathrm{erfc}(b\sqrt{2}) \,. \tag{24}$$

Collecting all expressions we obtain

$$⨍ d\mu(J) = K_N \prod_{i>j} |\lambda_i-\lambda_j| \prod_{i=1}^{R} \exp(-\lambda_i^2/2) \prod_{j=1}^{Q} \exp[-(a_j^2-b_j^2)]\mathrm{erfc}(b_j\sqrt{2}) d^{(R,Q)}\lambda \,, \tag{25}$$

with

$$K_N = \Omega(N) 2^{-N} (2\pi)^{-N(N+1)/4} \,. \tag{26}$$

Here ⨍ denotes the integration over all eigenvectors, which we have carried out. The left-hand side of (25) yields by definition the joint probability density of eigenvalues,

$$⨍ d\mu(J) = P(\lambda_1,\lambda_2,\ldots,\lambda_N) d^{(R,Q)}\lambda \,. \tag{27}$$

Equations (25)-(27) are the central result of our paper. It is possible to write $P(\lambda_1,\lambda_2,\ldots,\lambda_N)$ in a symmetric fashion:

$$P(\lambda_1,\lambda_2,\ldots,\lambda_N)=K_N\prod_{i>j}|\lambda_i-\lambda_j|\left\{\prod_{i=1}^{N}\exp(-\lambda_i^2)\operatorname{erfc}(|\lambda_i-\lambda_i^*|/\sqrt{2})\right\}^{1/2}. \tag{28}$$

Note, however, that the explicit form of the measure $d^{(R,Q)}\lambda$ depends on R and the chosen order (11). That means we have to distinguish between cases with $R=0$, $1,2,\ldots$ real eigenvalues and the rest of the eigenvalues being complex conjugate in pairs. Only summing over all R yields total probability 1. The prefactor $\prod_{i>j}|\lambda_i-\lambda_j|$ determines the degree of level repulsion. For a pair of adjacent real eigenvalues it is proportional to $|\lambda_i-\lambda_j|$, showing linear level repulsion as in the Gaussian orthogonal ensemble of real symmetric matrices. For two eigenvalues λ_1,λ_2 meeting in the upper complex plane, $\prod_{i>j}|\lambda_i-\lambda_j|$ is proportional to $|\lambda_1-\lambda_2|^2$, yielding together with the two-dimensional volume element the cubic level repulsion as in Ginibre's complex ensemble. Furthermore, for large N most of the eigenvalues lie far from the real axis; in this case the weight factor in (25) contributes for each eigenvalue in the upper plane a factor $\exp(-|\lambda_i|^2)$, again as in the complex ensemble. Because of this fact and with the help of Mehta's method [1] we are able to recover the circle law for the average eigenvalue density in the large-N limit.

In conclusion, we have derived the joint probability density of eigenvalues of asymmetric real matrices with a Gaussian measure, invariant under orthogonal transformations. Formulas (27) and (28) have to be interpreted for a given order of eigenvalues (11). Each case $R=1,2,\ldots,N$ has to be considered separately. For example, for N odd and $R=1$, expression (28) integrated with restriction (11) yields the probability that exactly one eigenvalue is real and all further eigenvalues are complex conjugate in pairs. We have a simple algebraic algorithm to calculate all these probabilities; e.g., for $N=2$ the probability that both eigenvalues are real is $1/\sqrt{2}$. We calculated the averaged fraction of real eigenvalues up to $N=60$ and found that it scales roughly as $\propto 1/\sqrt{N}$ for large N in agreement with Ref. [5]. It is remarkable that K_N does not depend on R.

In the limit of large N we could recover the circle law for the eigenvalue density in the completely asymmetric case ($\tau=0$). For the correlated case ($\tau\neq 0$), which is partly symmetric, we find the elliptic law, of course. For completeness and use in the context of neural networks we give here the joint probability density $P_\tau(\lambda_1,\lambda_2,\ldots,\lambda_N)$ for general covariances $\langle J_{ij}^2\rangle=1/N$, $\langle J_{ij}J_{ji}\rangle=\tau/N$, and $\langle J_{ii}^2\rangle=(1+\tau)/N$:

$$P_\tau(\lambda_1,\lambda_2,\ldots,\lambda_N)=D\exp\left[\tau\sum_i(\gamma\lambda_i)^2/2\right]P(\gamma\lambda_1,\gamma\lambda_2,\ldots,\gamma\lambda_N) \tag{29}$$

with

$$\gamma^2=N/(1-\tau^2)\ \text{and}\ D=[N/(1+\tau)]^{N/2}(1-\tau^2)^{N(N-1)/4}. \tag{30}$$

Our next task is to calculate all correlation functions, which will have a quite complicated structure due to the singular behavior on the real axis.

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