"Fractional Statistics" in Arbitrary Dimensions: A Generalization of the Pauli Principle

F. D. M. Haldane[a]
Department of Physics, Princeton University, Princeton, New Jersey 08544
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The concept of "fractional statistics" is reformulated as a generalization of the Pauli exclusion principle, and a definition independent of the dimension of space is obtained. When applied to the vortexlike quasiparticles of the fractional quantum Hall effect, it gives the same result as that based on the braid group. It is also used to classify spinons in gapless spin-$\frac{1}{2}$ antiferromagnetic chains as semions. An extensive one-particle Hilbert-space dimension is essential, limiting fractional statistics of this type to topological excitations confined to the interior of condensed matter. The new definition does not apply to "anyon gas" models as currently formulated: A possible resolution of this difficulty is proposed.

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The concept of "anyons" or particles with "fractional statistics" in two-dimensional (2D) systems [1] has been a subject of intense study in recent years, and has found application in the theory of the fractional quantum Hall effect [2] (FQHE). More recently, it has formed the basis of the theory of "anyon superconductivity" [3]. The anyon concept is essentially two dimensional; however, a recent study [4] of "spinon" excitations in *one*-dimensional antiferromagnets has led me to a variant notion of fractional statistics which, like the conventional Bose and Fermi statistics, is formulated without specific reference to spatial dimension.

This new definition can be considered as a generalization of the Pauli exclusion principle; when applied to the FQHE, it coincides with the now-standard 2D anyon definition [1] in terms of the braiding of particle trajectories; in general, however, the two definitions are *not* equivalent. The definition proposed here requires that single-particle Hilbert-space dimensions are *extensive,* which is a common property of, e.g., topological excitations of a condensed-matter state, but not of the flux-carrying Newtonian point particles of the "anyon gas" model [1]. The statistics as defined here is *not* affected by the attachment to the particles of gauge fields which conventionally lead to "statistical transmutation" of long-distance, low-energy properties: For example, "hard-core lattice bosons" would be classified as fermions (that carry a gauge field), despite their bosonlike low-energy properties, because of their fermionlike exclusion principle and band-filling property.

Consider the Hilbert space $\mathcal{H}_\alpha$ of states of a single particle of species α, confined to a finite region of matter, where this space is spanned by wave functions $\phi_v^\alpha(r)$, $v=1,\ldots,d_\alpha$. It will be crucial for this discussion that the dimension $\dim[\mathcal{H}_\alpha]=d_\alpha$ is *finite and extensive,* proportional to the size of the condensed-matter region in which the particle exists. This implies that the "particle" is an *elementary excitation that can only exist in the interior of a region of condensed matter,* and is not an elementary particle which can have arbitrarily large momentum, and exist in the vacuum outside the condensed-matter region. In general, the fractional-statistics particle will be a topological excitation of a condensed-matter state.

Now consider the wave function of an N-particle system of such particles with coordinates and species indices $\{r_i,\alpha_i;\, i=1,\ldots,N\}$ and let N_α be the number of particles of species α. If the coordinates of the $N-1$ particles with labels $j\neq i$ are held fixed, the wave function $\Psi(r_1,\ldots,r_N)$ can be expanded in a basis of wave functions of the ith particle as

$$\sum_v A_v(\{r_j,\alpha_j;j\neq i\})\,\phi_v^{\alpha_i}(r_i;\{r_j,\alpha_j;j\neq i\})\,. \qquad (1)$$

The set of wave functions $\phi_v^\alpha(r;\{r_j,\alpha_j;j\neq i\})$ span a one-particle Hilbert space $\mathcal{H}_\alpha(\{r_j,\alpha_j;j\neq i\})$, with dimensions $d_\alpha(\{N_\alpha\})$. This dimension must be independent of the *coordinates* of the particles with labels $j\neq i$, and must be the same for all identical particles of the same species. It will depend only on the boundary conditions and size of the condensed-matter region and the numbers $\{N_\alpha\}$ of the different particle species.

In general, d_α will change as particles are added, *while keeping the boundary conditions and size of the condensed-matter region constant.* This will provide the basic notion of statistics developed here. I define the *statistical interaction* $g_{\alpha\beta}$ through the differential relation

$$\Delta d_\alpha = -\sum_\beta g_{\alpha\beta}\Delta N_\beta\,, \qquad (2)$$

where $\{\Delta N_\beta\}$ is a set of allowed changes of the particle numbers at fixed size and boundary conditions. Conventional bosons have $g_{\alpha\beta}=0$, while the Pauli exclusion principle for fermions corresponds to $g_{\alpha\beta}=\delta_{\alpha\beta}$. The *relation (2) may be considered as a generalization of the Pauli principle.*

In order for a thermodynamic limit to exist with extensive single-particle Hilbert-space dimensions d_α, the statistical interactions $g_{\alpha\beta}$ must be independent of the numbers of particles. The existence of a thermodynamic limit also requires that the $g_{\alpha\beta}$ are *rational,* so that limit can be achieved through a sequence of proportional finite increments of the size of the system and the particle numbers.

For state-counting purposes *at fixed particle numbers* the particles can be regarded as bosons with a Fock-space dimension $d_B = d_\alpha$ or fermions with a Fock-space dimension $d_F = d_\alpha + N_\alpha - 1$. In either interpretation, the total size of the full Hilbert space of many-particle states, at fixed $\{N_\alpha\}$, will be

$$\prod_\alpha \frac{(d_\alpha + N_\alpha - 1)!}{(N_\alpha)!(d_\alpha - 1)!} . \tag{3}$$

The construction of a fermionlike or bosonlike description of the states thus does *not* imply anything about the statistics: The particles are only true bosons (fermions) if their effective Fock-space dimension d_B (d_F) *remains constant as the number of particles is changed.*

Unless this condition is satisfied, the conventional techniques of second-quantized many-body theory cannot be applied. Arguably, the key step in Laughlin's [5] seminal treatment of the FQHE, so far the only established physical application of fractional statistics [2], was to abandon conventional second-quantized methods, which had proved fruitless, and return to a first-quantized description.

In the FQHE, the Pauli-like definition of statistics introduced here can be related to the braid-group notion of 2D statistics [1]. The basic model for 2D anyons is the charged flux tube [1]. I will use units where $e = \hbar = c = 1$, in which the London flux quantum Φ_0 is 2π. If an object carrying flux ϕ_1 and charge q_1 orbits around another object carrying flux ϕ_2 and charge q_2, the Bohm-Aharonov phase change of the wave function is

$$\exp(i\theta_{12}) = \exp[-i(q_1\phi_2 + q_2\phi_1)] . \tag{4}$$

If they are identical, the Bohm-Aharonov phase factor for interchange in $\exp(i\theta_1) \equiv \exp(i\theta_{11}/2)$. Anyons with an aribitrary statistical parameter θ_α are modeled by bosonic flux tubes with $q_\alpha \phi_\alpha = \theta_\alpha$.

In the FQHE, the quasiparticles of the Laughlin states have the character of vortices [5], with dynamics derived from quantizing the Eulerian dynamics of point vortices in an incompressible, inviscid fluid (which do not carry kinetic energy), rather than the Newtonian dynamics of a massive particle. The "guiding center" coordinates of the vortex do not commute:

$$[R_i^\mu, R_j^\mu] = il^2 \delta_{ij} q_i \epsilon^{\mu\nu} . \tag{5}$$

Here $2\pi l^2$ is the area per particle of the underlying fluid; the integer q_i is the circulation of the ith vortex in units of the elementary circulation quantum of the fluid.

The commutation relations (5) imply [6] that the vortex cannot be localized in an area smaller than the mean area per fluid particle, and states representing the vortex centered at different point are *nonorthogonal*; for a fluid with open boundary conditions, the number of independent states of a vortex is $N+1$, where N is the number of particles in the underlying fluid. These particles act as quantized sources of "flux" ($\phi = 2\pi$ per particle); (5) implies that if an elementary quantum vortex carrying "charge" ± 1 moves around a closed path, its wave function picks up a phase factor that counts the mean number of particles enclosed by that path. (This complements the result that a fluid particle moving around a closed loop picks up a phase that counts the total vorticity inside the loop, forcing quantization of the circulation around a vortex [6].) The model for a fractional-statistics vortex is one which locally has a mean excess or deficit of fluid particles as compared to the mean fluid density in its absence. This is only well defined in an incompressible fluid [6].

The noncommutativity of guiding-center coordinates makes vortex wave functions equivalent to the *lowest-Landau level states of a charged particle in a magnetic field.* The FQHE quasiparticles can thus be modeled by flux-carrying charged bosons in the lowest Landau level. If there is a total flux Φ through the system, the number of independent single-particle states in the Landau level for bosonic particles carrying charge q_α is $d_\alpha = q_\alpha \Phi/2\pi$. If the particles carry flux ϕ_α, the total flux Φ, and hence the $\{d_\alpha\}$, change as particles are added, and $g_{\alpha\beta} = -q_\alpha \phi_\beta/2\pi$. The relative statistical phases $\theta_{\alpha\beta}$ for windings of particle trajectories are thus identified as $\exp(i\theta_\alpha) = \exp(\zeta i\pi g_{\alpha\alpha})$, and $\exp(i\theta_{\alpha\beta}) = \exp[\zeta i\pi(g_{\alpha\beta} + g_{\beta\alpha})]$. (An overall handedness $\zeta = \pm 1$ remains undefined.) Note that the statistical interactions $g_{\alpha\beta}$ convey more information than the statistical phases, which are ambiguous modulo 2π.

The Laughlin FQHE states at the primary Landau-level fillings $\nu = 1/m$ have two vortexlike excitations, quasiparticles and quasiholes. In this context, fixing the boundary conditions means fixing $BA/\Phi_0 = N_\Phi$, the total magnetic flux (in units of the flux quantum) passing through the system. Let there be N^+ quasiparticles and N^- quasiholes: The number of electrons N is then given by [7] $N_\Phi = m(N-1) + N^- - N^+$. The Hilbert-space dimension $d_\pm$ for both quasiparticles and quasiholes is [7] $N+1$. Changing N^+ and N^- by multiples of m at fixed N_Φ gives $g_{\alpha\pm} = \mp 1/m$, $\alpha = \pm$. The agreement between the Hilbert-space counting definition of statistics and the anyon definition in this case is no accident: Like the statistical phase, the Hilbert-space dimension can be obtained from the Bohm-Aharonov-like Berry's phase for adiabatic transport of a quasiparticle around a loop, in this case one encircling the entire fluid.

The dimension-independent definition (2) of fractional statistics opens up possibilities of non-2D applications. In principle, fractional statistics would be recognized by the presence of bands with an unusual (and variable) number of single-particle states, not given by simply counting the number of unit cells or atoms in the condensed-matter system.

As an example not restricted to two spatial dimensions, I consider spinon excitations in a spin-$\frac{1}{2}$ quantum antiferromagnet with a *nondegenerate* singlet "resonating valence bond" (RVB) ground state [8] without magnetic order: These may be thought of as isolated unpaired spins [9] in a RVB background of paired spins. The spinon is a spin-$\frac{1}{2}$ object coming in two species, labeled by $\sigma = \pm\frac{1}{2}$. If there are N spins and N_{sp} spinons, the number of unbroken bonds is $(N-N_{sp})/2$, which must be an integer. I identify the spinon Hilbert-space dimension d_σ as $1+(N-N_{sp})/2$, independent of σ.

This can be understood as follows: A given spinon can occupy the site it is initially on, or it can be moved to a site that is part of a pair. But if $|1(23)\rangle$ represents a three-site wave function where site 1 is unpaired and sites 2 and 3 are paired, nonorthogonality means that $|1(23)\rangle = (1/\sqrt{2})[|2(13)\rangle - |3(12)\rangle]$. Hence only the combination $[|2(13)\rangle + |3(12)\rangle]$ is independent of $|1(23)\rangle$. There is thus only *one* extra independent spinon state per bond. *This reduction of apparent Hilbert-space dimension by nonorthogonality of states describing localized topological defects at different points in space is also seen in the FQHE example, and seems to be the fundamental feature of fractional statistics as defined here.*

The statistical interaction between spinons is thus given by $g_{\sigma\sigma'}=\frac{1}{2}$, independent of spin. It can now be verified that the full Hilbert space of the spin system is spanned by the semionic many-spinon states, with no over or under counting. The total number of states is obtained using (3). At a given spinon number, the number of many-spinon states is equivalent to the number of ways to place $N_{sp}=N_{sp\uparrow}+N_{sp\downarrow}$ bosons in $2\times[1+(N-N_{sp})/2]$ orbitals; taking the background RVB state to be nondegenerate [10], the full Hilbert-space dimension is

$$\frac{1}{2}\sum_{N_{sp}}[1+(-1)^{(N-N_{sp})}]\frac{(N+1)!}{N_{sp}!(N-N_{sp}+1)!}. \tag{6}$$

This is 2^N, as expected, showing completeness of the spinon description. Note that this discussion has not involved properties of the Hamiltonian, except perhaps through the assumption of a nondegenerate ground state. However, the spinon description will only be useful if the elementary excitations do indeed have spinon character.

Explicit examples of the above scenario are provided by spin-$\frac{1}{2}$ antiferromagnetic Heisenberg spin chains, in the gapless phase which at low energies is described by the level $k=1$ SU(2) Wess-Zumino-Witten conformal field theory (CFT) [11]. One example is the nearest-neighbor-exchange Heisenberg chain solved by Bethe [12]; however, the clearest and most explicit example is provided by the $S=\frac{1}{2}$ Heisenberg chain with inverse-square exchange [4,13,14] (ISE model), which also generates the $k=1$ CFT. The ISE model has the same state-counting rules as Bethe's model, but has a much simpler spectrum, allowing its thermodynamics to be constructed explicitly in closed form [4]. It is also deeply related to Laughlin's $\nu=\frac{1}{2}$ boson FQHE state [5], making a fractional-statistics interpretation very natural.

If N is the (even) number of sites, the spin-singlet ground-state wave function of the ISE model can be written in two forms [13] directly related to Laughlin FQHE wave functions [5]: The first form (expressed in terms of the $N/2$ sites $\{n_i\}$ "occupied" by reversed spins) is the Laughlin-Kalmeyer form [15], now defined on a 1D rather than a 2D lattice. This is a direct transcription of the $\nu=\frac{1}{2}$ bosonic FQHE wave function. If $z(n) = \exp(2\pi i n/N)$,

$$\Psi=\prod_{i<j}[z(n_i)-z(n_j)]^2\prod_i z(n_i). \tag{7}$$

The other form is the $n=2$ case of the SU(n) singlet wave function where $\langle\sigma_1,\ldots,\sigma_N|\Psi\rangle$ vanishes unless $\sum\delta_{\sigma,\sigma_n}=N/n$, when it obtains from the antisymmetric Slater determinant function $\Psi(\{z_m,\sigma_m\})$ given by

$$\prod_{m<n}(z_m-z_n)^{\delta(\sigma_m,\sigma_n)}(i)^{\mathrm{sgn}(\sigma_m-\sigma_n)}, \tag{8}$$

with "spatial coordinates" $\{z_m\}$ chosen to be a permutation of $\{z(m),\ m=1,\ldots,N\}$. Up to a spin-independent factor, (8) is the ground state of SU(n) fermions in 1D or filling the lowest Landau level in 2D [16].

The great simplifying feature of the ISE model is that there are *no* spin-dependent interactions between the spinons [4]. The spinon states form a band of $1+(N-N_{sp})/2$ states with a spin-independent statistical interaction $g_{\sigma\sigma'}=\frac{1}{2}$, just as predicted by the general argument given here. For a fixed N_{sp}, a spin-$\frac{1}{2}$ bosonic Fock space is the most appropriate description: The eigenstates are characterized by sets of occupation numbers, just as in the case of the ideal Fermi or Bose gases, except that the energy is a quadratic (instead of linear) function of the occupations. The full solution [4] for the thermodynamics of the ISE model shows that occupation-number distributions *in a Fock space that varies with temperature as the number of fractional-statistics excitations changes* have a role to play in the theory of fractional-statistics systems.

Because of spin exchange between spinons (which is marginally irrelevant at low energies) the Bethe-ansatz model has a much more complicated solution; however, from a study [17] of the adiabatic interpolation between the two models, I have established that the total number of complex rapidity strings in the Bethe-ansatz solution *(irrespective of their length)* has the simple interpretation as the number of unbroken valence bonds, $(N-N_{sp})/2$.

The ISE-model spinon band covers half the Brillouin zone (BZ), shrinks as spinons are added, and is gapless at its termination points. If an analogous RVB ground state can occur in some higher-dimensional model *without symmetry breaking that reduces the BZ volume, or leads to a degenerate ground state* [10], the result for d_σ obtained here requires the existence of a surface enclosing half the BZ volume which would mark the gapless boundary of the spin band. (The normalization of Bloch-state combinations of localized spinon states will diverge as this boundary is approached from the interior.) This scenario resembles that of the "pseudo-Fermi surface" which Anderson has predicted [8] to characterize a gapless 2D RVB state. The ideas developed here suggest that gapless Fermi-surface-like structures, and a generalized, *fractional* Luttinger-type relation between volume enclosed by such surfaces and quasiparticle number, may be a new type of collective behavior in two or more dimensions.

In the examples discussed so far, the fractional statistics emerges in the properties of elementary excitations of models built microscopically out of conventional objects such as electrons. In the anyon gas model [1], a Chern-Simons gauge field is added "by hand" to a model of conventional particles to produce Bohm-Aharonov phases as the particles orbit each other. A lattice-gas version of the model which satisfies the Hilbert-space extensivity condition consists of hard-core bosons (or spinless fermions) to which Chern-Simons flux is attached [18]. Since addition of a coupling to a gauge field does not affect Hilbert-space dimensions, such particles are classified as fermions by the definition proposed here.

The difference between the two definitions of statistics in 2D—when applied to the anyon gas model—is troubling: However, I note that the model has no true microscopic derivation. A possible resolution of the discrepancy may be conjectured: The appearance of a Chern-Simons field in the effective low-energy description of topological excitations of a 2D condensed-matter state of conventional particles may *inevitably* (as in the FQHE) be accompanied by nonorthogonality of localized states of the topological excitations, which reconciles the two definitions.

The ingredients of a modified "lattice anyon" model with an analog of Euler dynamics (appropriate to vortices defined on the plaquettes of a dual lattice) can be identified: If i is a "site" (plaquette) label, they are (a) a site-diagonal one-body Hamiltonian $H=\epsilon_0\sum_i|i\rangle\langle i|$ and (b) *nonorthogonality,* $\langle i|j\rangle=S_{ij}\neq\delta_{ij}$, which replaces kinetic energy as the generator of dynamics. The Hermitian overlap matrix S_{ij} has real non-negative eigenvalues; the number of states in the band is the number of *nonzero* eigenvalues S_ν, and the corresponding eigenvalues of the one-body Hamiltonian are $\epsilon_0 S_\nu$. If the rank of the overlap matrix is less than its dimension, the number of eigenstates of H is less than the number of sites. In the many-body case, the overlap matrix for a given particle (i.e., vortex) would depend on the number and positions of other particles, introducing both gauge interactions and fractional statistics as defined here.

In summary, I have introduced a variant definition (2) of fractional statistics that can be viewed as a generalization of the Pauli principle, and does not make reference to spatial dimension. It produces consistent results when applied to 2D Laughlin FQHE quasiparticles and spinons in nondegenerate RVB states of 1D $S=\frac{1}{2}$ quantum antiferromagnets. However, it does *not* apply to 2D models where Chern-Simons flux has been attached "by hand" to conventional particles. Such models lack what appears to be an essential element of fractional statistics as defined here: nonorthogonality of localized particle states.

Finally, I note that temperature appears to play no role in this approach, in contrast to a recent proposal [19] in which "statistics" is identified with "local fermionic charge."

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(a)On leave of absence from Department of Physics, University of California, San Diego, La Jolla, CA 92093.

[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento **37B**, 1 (1977); F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).
[2] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); **52**, 2390(E) (1984).
[3] R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).
[4] F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).
[5] R. B. Laughlin, Phys. Rev. Lett. **50**, 1359 (1983).
[6] F. D. M. Haldane and Y.-S. Wu, Phys. Rev. Lett. **55**, 2887 (1985).
[7] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[8] P. W. Anderson, Science **235**, 1196 (1987).
[9] S. A. Kivelson, D. S. Rokhsar, and J. R. Sethna, Phys. Rev. B **35**, 8865 (1987).
[10] This applies to the ISE model of Refs. [13,14], but not to a dimerized valence bond state, nor to the chiral 2D state of Ref. [15] which (with periodic boundary conditions) is doubly degenerate in *both* chiral sectors.
[11] I. Affleck, Nucl. Phys. **B265 [FS15]**, 409 (1986).
[12] H. A. Bethe, Z. Phys. **71**, 205 (1931).
[13] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).
[14] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
[15] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[16] *Note added.*—Since submission of this Letter, I have found that this construction also gives the exact ground state of the SU(n) generalization of the ISE model [17].
[17] F. D. M. Haldane (unpublished).
[18] G. S. Canwright, S. M. Girvin, and A. Brass, Phys. Rev. Lett. **63**, 2291 (1989).
[19] P. C. E. Stamp, Europhys. Lett. **14**, 569 (1991).