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Quasimonochromatic Noise: New Features of Fluctuations in Noise-Driven Nonlinear Systems

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The effects of Gaussian quasimonochromatic (narrow band) noise have been investigated with an analog electronic circuit model. The escape probability and the reciprocal mean time to reach the top of a potential barrier were found to differ exponentially strongly. An extremely steep statistical distribution was observed and is discussed. For systems fluctuating in a symmetric single-well potential the distribution was found to be independent of the shape of the potential.

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During the last few years substantial progress has been achieved towards an understanding of fluctuation phenomena in nonlinear systems driven by colored noise [1]. The problem is interesting not only because of its immediate relevance to numerous particular systems, including dye lasers and liquid crystals, but also from a more general point of view, since the white-noise approximation that goes back to Einstein and Smoluchowski is an idealization. It is important therefore to find out in which cases the existence of structure in the power spectrum of the noise does change qualitatively the form of the fluctuations induced in a system, and what are the corresponding changes in such fundamental characteristics as the shape of the statistical distribution and the pattern of motion leading to noise-induced escape from a stable state.

It is well known [2], in particular, that a system driven by white noise (a Brownian particle) escapes from a stable state via fluctuations bringing it slightly over (for weak noise) the potential-barrier top (PBT). In other words, to an accuracy of a prefactor, the mean first passage time (MFPT) from near the bottom of the potential well to PBT, t_{top}, is equal to the reciprocal escape probability W^{-1}. The same notion has also been applied in numerous papers (e.g., Refs. [1], [3], and [4], and references therein) to colored noise having a power spectrum in the form of a Lorentzian peak centered at zero frequency, the model of colored noise investigated in most detail so far.

We demonstrate in this Letter that a colored-noise–driven bistable system can pass the PBT several times, back and forth, but nonetheless still return to the initially occupied stable state with an overwhelming probability, instead of making a transition to the other stable state. The MFPT to the PBT may consequently be less by several orders of magnitude than the reciprocal escape (transition) probability, $t_{\text{top}} \ll W^{-1}$. The particular noise, $f(t)$, investigated to find this general (for colored noise [5]) effect was a zero-mean Gaussian quasimonochromatic noise [6] (QMN) with a power spectrum $\Phi(\omega)$ having a narrow Lorentzian peak centered, not at zero frequency, but at a finite frequency ω_0,

$$\Phi(\omega) = \int_{-\infty}^{\infty} dt \exp(i\omega t)\langle f(t)f(0)\rangle$$
$$= 4\Gamma T/[(\omega^2-\omega_0^2)^2+4\Gamma^2\omega^2], \quad \Gamma \ll \omega_0 . \qquad (1)$$

QMN corresponds to the thermal noise of an underdamped harmonic oscillator with the eigenfrequency ω_0 and damping coefficient $\Gamma \ll \omega_0$ coupled to a bath with temperature T (in Refs. [6(b)] and [6(c)], noise with the spectrum (1) was called "harmonic"). It is QMN that gives rise to the fluctuations in a variety of physical systems, for example, in those coupled to fluctuating (e.g., thermal) high-Q electromagnetic or acoustic intracavity modes, or to impurities performing localized or resonant vibrations in solids, or to eigenvibrations of large molecules or engineering structures. As is demonstrated below, not only is there a dramatic difference between t_{top} and W^{-1}, but there are also some very specific features of the statistical distribution [5] that are innate to QMN-driven systems.

We consider overdamped dynamics in a potential

$U(x)$,

$$\dot{x}+U'(x)=f(t). \quad (2)$$

The stable equilibrium points x_i of the system correspond to the minima of $U(x)$, at $U'(x)=0$; and its relaxation time $\tau=\max[U''(x_i)]$. To gain insight into the characteristic features of the fluctuations of the dynamical variable x we note that, according to (1), the QMN $f(t)$ can be viewed as a superposition of nearly periodic random vibrations at frequency ω_0, with the correlation time of their amplitude and phase equal to Γ^{-1}, and a small nonresonant addition $\delta f(t)$,

$$f(t)=f_+(t)\exp(i\omega_0 t)+f_-(t)\exp(-i\omega_0 t)+\delta f(t),$$
$$\langle|\dot{f}_\pm|^2\rangle/\langle|f_\pm|^2\rangle\sim\Gamma^2\ll\omega_0^2,\quad \langle(\delta f)^2\rangle\ll\langle|f_\pm|^2\rangle. \quad (3)$$

Some remarkable physical effects arise when the reciprocal relaxation time of the system τ^{-1} lies between Γ and ω_0:

$$\Gamma\ll\tau^{-1}\ll\omega_0. \quad (4)$$

It follows from (2)-(4) that to the lowest order in $\Gamma\tau$, $(\omega_0\tau)^{-1}$,

$$x(t)=x_+(t)\exp(i\omega_0 t)+x_-(t)\exp(-i\omega_0 t)+x_c(t), \quad (5)$$
$$x_\pm(t)\simeq(\pm i\omega_0)^{-1}f_\pm(t),\quad x_c\simeq x_c^{(\mathrm{ad})}(x_+,x_-).$$

Here, $x_c^{(\mathrm{ad})}(x_+,x_-)$ is the equilibrium position of the center of the forced vibrations of the coordinate $x(t)$ with a given amplitude $2|x_+|$. It is given by

$$V_c'(x_c^{(\mathrm{ad})},x_+,x_-)=0,\quad V_c'\equiv\partial V/\partial x_c,$$
$$V\equiv V(x_c,x_+,x_-) \quad (6)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}d\psi\, U(x_c+x_+e^{i\psi}+x_-e^{-i\psi}).$$

In obtaining (5) we have neglected $\delta f(t)$ in (3) and allowed for $x_\pm(t)\propto f_\pm(t)$ being smoothly varying [cf. (3)] over times $\sim\tau$, so that $x_c(t)$ follows $x_\pm(t)$ adiabatically. $V(x_c,x_+,x_-)$ is the potential $U(x)$ averaged over time $\sim\omega_0^{-1}$.

Equation (5) enables one to understand the origin of the difference between the reciprocal MFPT to the PBT from a stable state i, and the probability W_{ij} of the $i\to j$ transition. It is clear [see Fig. 1(a)] that the oscillating $x(t)$ will reach the position x_{top} (the PBT) for the first time with increasing $|f_+(t)|$ for $x_c^{(\mathrm{ad})}(t)$ lying on the x_i side of x_{top}, where x_i is the initially occupied position. To bring $x_c^{(\mathrm{ad})}$ to x_{top}, still higher values of $|f_+|$ are necessary. But fluctuations giving rise to such $|f_+|$ are precisely those needed for an interwell transition: The transition can occur, without additional forcing, once x_c reaches x_{top}, while in the opposite case the system will go back to the initially occupied state as $|f_+|\propto|x_+|$ dies out, according to (5). Therefore, $W_{ij}t_{\mathrm{top}}\ll 1$.

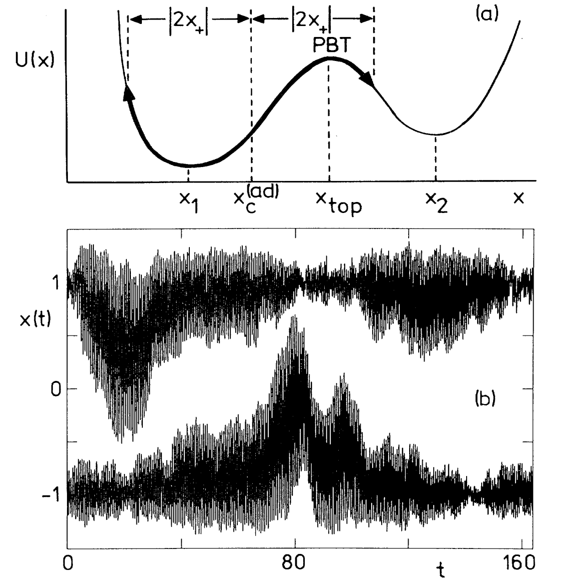

FIG. 1. (a) Sketch of QMN dynamics in a potential $U(x)$, illustrating the discussion of Eq. (5). The coordinate oscillates (thick line with arrows) with amplitude $|2x_+|$ about a center of motion $x_c^{(\mathrm{ad})}$, and can pass PBT on each cycle without making a transition out of the initially occupied potential well. (b) Two samples of $x(t)$ measured with $D=192$ for the analog electronic circuit model (1), (2), and (7), exhibiting an example of an occasional large fluctuation from each of the attractors.

These arguments have been tested with the aid of an electronic analog model of (1) and (2) of conventional design [7]. The bistable potential $U(x)$ was chosen to be of the widely used [1] form

$$U(x)=-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4\quad(x_1=-1,\ x_2=1,\ x_{\mathrm{top}}=0). \quad (7)$$

QMN was obtained from the electronic analog of an underdamped harmonic oscillator with $\omega_0=9.81$, $\Gamma=0.021$ that was driven by the pseudo white noise from a feedback shift-register noise generator.

Two samples of the fluctuating voltage in the circuit representing $x(t)$ in (2) are shown in Fig. 1(b). They correspond to fluctuations about the stable states $x_1=-1$ and $x_2=1$ and each exhibits occasional large fluctuations. It is apparent that in the course of such fluctuations $x(t)$ crosses the boundary point $x_{\mathrm{top}}=0$ several times forwards and backwards, and then goes back to the initially occupied state without making a transition. Paths resulting in transitions to the other state are very much less frequent; they have been observed to happen for larger amplitudes of fluctuational vibrations and, correspondingly, larger shifts of their centers, in full agreement with the above qualitative picture.

Since $x=0$ is thus an "ordinary" value of x from the viewpoint of intrawell fluctuations, the quasistationary statistical distribution $p_i(x)$ about the initially occupied position x_i [where $p_i(x)$ is formed over the times $\Gamma^{-1}, \tau \ll t \ll W_{ij}^{-1}$] would not be expected to have an extremum at $x=0$, in striking contrast to the case of the white-noise forcing. This has been confirmed by direct measurements of $p_i(x)$; see Fig. 2.

For sufficiently small noise intensities $D=T/\Gamma(\omega_0^2-\Gamma^2)$ and large $|x-x_i|$, the quantity

$$R_i(x)=D|\ln[p_i(x)/p_i(x_i)]|$$

plotted in Fig. 2 gives the activation energy of the reciprocal MFPT to a given x from x_i, and, in particular, $t_{top} \sim \Gamma^{-1}\exp[-R_i(0)/D]$. [Note that a "small" noise intensity implies $D \ll R$, where (see below) $R \sim \omega_0^2/\Gamma$. Because Γ is itself very small, "small" values of D may correspond to quite large numbers.] In agreement with the qualitative observations (cf. Fig. 1) the value $R_i(0)=910$ obtained for $\omega_0=9.81$, $\Gamma=0.021$ was substantially less than the activation energy $R_{it}=-D\ln(W_{ij}/\Gamma)$ of the interwell transitions for the same parameters, $R_{it}=1480$ (R_{it} was obtained from the values of $\ln W_{ij}$ for several D^{-1} that lay on a straight line to a good accuracy). Both $R_i(0)$ and R_{it} are very close to the theoretical predictions [5] $\omega_0^2/5\Gamma$ and $\omega_0^2/3\Gamma$, respectively. For the lowest D investigated R_{it}/D and $R_i(0)/D$ were equal to 11.2 and 6.89, respectively, thus demonstrating that W_{ij} and t_{top}^{-1} indeed differed by a large factor $\sim 10^2$.

One more qualitative feature obvious from Fig. 1(b) is that fluctuations about x_i are strongly "asymmetric": The system moves from x_i towards x_{top} much more "eagerly" than in the opposite direction. Correspondingly, the quasistationary statistical distribution $p_i(x)$ would be expected to decrease extremely sharply with increasing $|x-x_i|$ for $(x-x_i)/x_i>0$. This is seen from Fig. 2 to be the case in reality. This phenomenon can be readily understood if one notices that in the case of the potential

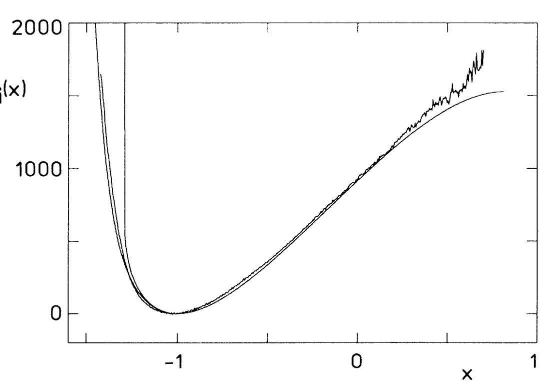

FIG. 2. The normalized quasistationary distribution (activation energy of the reciprocal MFPT from x_i to x), $R_i(x)=D|\ln[p_i(x)/p_i(x_i)]|$, $i=1$. Data measured (jagged line) for the analog circuit model of (1), (2), and (7) for $D=189$ are compared with the zeroth-order theory (solid curve, singular at $x \simeq -1.29$) and the first-order theory (the other solid curve).

(7), the center $x_c^{(ad)}$ of the vibrations of $x(t)$ shifts as $(-1)^i(1-6|x_+|^2)^{1/2}$ with the varying vibration amplitude $2|x_+|$. Therefore, the limiting value of the coordinate $x_c^{(ad)} \pm 2|x_+|$ does not overcome $(-1)^i x_0$ ($x_0=\sqrt{5/3}$) in the adiabatic approximation (5), so that $p_i(x)$ would be discontinuous at this value of x.

To describe the statistical distribution as a whole, and its steep decrease for $|x|>x_0$ in particular, it is convenient to use [5,8] the method of optimal fluctuation in Gaussian-noise-driven dynamics [3(a),3(c),9,10] which is based substantially on the path-integral approach [11] to such dynamics. To logarithmic accuracy, the calculation of $p_i(x)$ amounts to the solution of the set of equations

$$\begin{aligned}
&\ln[p_i(x)/p_i(x_i)]=-R_i(x)/D, \quad D=T/\Gamma(\omega_0^2-\Gamma^2),\\
&R_i(x)\simeq \tfrac{1}{2}\int_{-\infty}^{\infty} dt\, f(t)F(-id/dt)f(t),\\
&F(\omega)=D/\Phi(\omega),\\
&F(-id/dt)f(t)=\lambda(t), \quad \dot\lambda(t)=U''(x)\lambda(t),\\
&x(-\infty)=x_i, \quad f(\pm\infty)=0, \quad \lambda(-\infty)=0,\\
&x(0)=x, \quad \lambda(t)=0 \text{ for } t>0,
\end{aligned} \tag{8}$$

where $x(t)$ and $f(t)$ are interrelated via (2).

To lowest order in $\Gamma\tau$ and $(\omega_0\tau)^{-1}$ the solution of (8) is given by (5), and $R_i(x)$ is discontinuous [5] at $x=(-1)^i x_0$. The discontinuity is smeared out primarily because of the sharp increase in $\delta f(t)$ as x approaches $(-1)^i x_0$: $\delta f \sim (\Gamma/\tau)^{1/2}$ for $|x-(-1)^i x_0| \sim (\Gamma\tau)^{1/2}$. Allowing for this increase (the details will be given elsewhere), one arrives at the expression

$$\begin{aligned}
&R_i(x)\simeq 2\omega_0^2\Gamma^{-1}[x_+^2+\tfrac{3}{64}(\delta f^2/\Gamma V''_{cc})],\\
&x=x_c^{(ad)}+2x_+ +\tfrac{3}{4}\delta f/V''_{cc},\\
&x_+=x_-=(\delta f/8\Gamma)[1-(V''_{c+}/V''_{cc})]-(\delta f^2/64\Gamma V''^2_{cc})\\
&\qquad\times(7V'''_{cc+}-5V'''_{ccc}-V'''_{c++}-V'''_{c+-}),
\end{aligned} \tag{9}$$

where the subscripts, $c, \pm$ mean differentiation of $V(x_c, x_+, x_-)$ with respect to $x_c, x_\pm$, respectively; all derivatives are evaluated at $x_c=x_c^{(ad)}$. It is obvious from (9) that $R_i'(x)/R_i(x) \sim (\Gamma\tau)^{-1/2}$ for $[(-1)^i x - x_0] \sim (\Gamma\tau)^{1/2}$, i.e., within the latter extremely small range of x the logarithm of the statistical distribution increases by an order of magnitude. Equation (9) is seen from Fig. 2 to be in both qualitative and rather good quantitative agreement with the experimental data (note that the data refer to $\Gamma\tau=0.042$, and thus the nonanalytic in $\Gamma\tau$ corrections to $R_i(x)$ for $[(-1)^i x - x_0] \sim (\Gamma\tau)^{1/2}$ omitted in (9) would be noticeable).

Peculiar features of the statistical distribution arise not only for bistable, but also for monostable QMN-driven systems. This happens [5] even in the simplest case of fluctuations in a symmetric single-well potential: $U(x)$

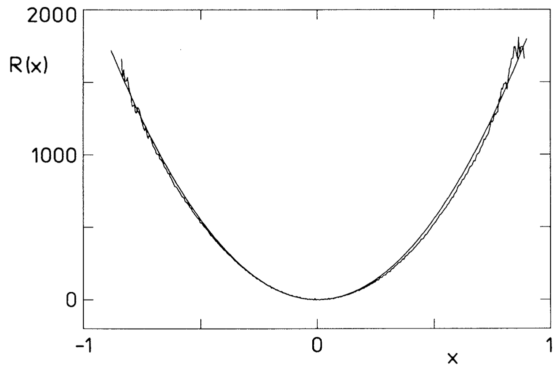

FIG. 3. The normalized statistical distribution $R(x) = D|\ln[p(x)/p(0)]|$ measured (jagged line) for an analog circuit model of (1) and (2) for $D = 160$ and the monostable potential $U(x) = \frac{1}{2}x^2$; measurements made for $U(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4$ and $U(x) = \frac{1}{6}x^2$ were found to be coincident. The smooth curve represents the exact theory (10) for the potential $U(x) = \frac{1}{2}x^2$.

$= U(-x)$, $U'(x)/x > 0$ in (2). It follows from Eqs. (5) and (6) that $x_c^{(\text{ad})} = 0$, while $x_\pm$ are independent of the particular shape of $U(x)$. The distribution $p(x)$ for sufficiently large $|x|$ is proportional to the probablity of the process $x(t)$ reaching $x = 2|x_+|$, i.e., to the probability of $|f_+(t)|$ reaching the value $\frac{1}{2}\omega_0|x|$ [see (5)]; thus *the distribution should be independent of $U(x)$* as well, $p(x) \propto \exp(-\omega_0^2 x^2/2\Gamma D)$. This remarkable invariance of $p(x)$—its independence both of the curvature at the potential minimum and of the nonparabolicity of the potential—is confirmed by the experimental data of Fig. 3 which are clearly in good agreement with the theory. Data measured for the harmonic potential $U(x) = \frac{1}{2}ax^2$ were in perfect agreement, within the experimental uncertainty of $\pm 2\%$, with the exact expression

$$p(x) = (A/2\pi)^{1/2}\exp(-\tfrac{1}{2}Ax^2)\,,$$
$$A = \frac{a\omega_0^2}{\Gamma D(\omega_0^2 - \Gamma^2)}\,\frac{(a^2+\omega_0^2) - 4\Gamma^2 a^2}{a(a^2+\omega_0^2) - 4\Gamma^2 a + 2\Gamma\omega_0^2}\,. \tag{10}$$

Equation (10) goes over into the above expression, $p(x) \propto \exp(-\omega_0^2 x^2/2\Gamma D)$, for $\Gamma \ll a \ll \omega_0$.

In conclusion, we emphasize the dramatic differences in the pattern of escape from a stable state for systems driven by QMN, on the one hand, and by white or exponentially correlated noise on the other. These differences, together with the singular features of the statistical distribution that have been observed, clearly demonstrate the possibility of controlling not only the intensity, but also the fundamental qualitative features of fluctuations in a system by varying the power spectrum of a driving noise.

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[1] See, for example, J. M. Sancho and M. San Miguel, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge Univ. Press, Cambridge, 1989), Vol. 1, p. 72; K. Lindenberg, B. J. West, and J. Masoliver, *ibid.*, p. 110; P. Hanggi, *ibid.*, p. 307; K. Lindenberg, B. J. West, and G. P. Tsironis, Rev. Solid State Sci. **3**, 143 (1989); H. R. Brand, C. R. Doering, and R. E. Ecke, J. Stat. Phys. **54**, 1111 (1989), and references overviewed therein.

[2] H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).

[3] (a) J. F. Luciani and A. D. Verga, Europhys. Lett. **4**, 255 (1987); J. Stat. Phys. **50**, 567 (1988); (b) G. P. Tsironis and P. Grigolini, Phys. Rev. Lett. **61**, 7 (1988); Phys. Rev. A **38**, 3749 (1988); F. J. de la Rubia, E. Peacock-Lopez, G. P. Tsironis, K. Lindenberg, L. Ramirez-Piscina, and J. M. Sancho, Phys. Rev. A **38**, 3827 (1988); (c) M. M. Klosek-Dygas, B. J. Matkowsky, and Z. Schuss, SIAM J. Appl. Math. **48**, 425 (1988); J. Stat. Phys. **54**, 1309 (1989); A. J. Bray and A. J. McKane, Phys. Rev. Lett. **62**, 493 (1989); A. J. Bray, A. J. McKane, and T. J. Newman, Phys. Rev. A **41**, 657 (1990).

[4] C. R. Doering, P. S. Hagan, and C. D. Levermore, Phys. Rev. Lett. **59**, 2129 (1987); **60**, 2805 (1988); R. F. Fox, Phys. Rev. A **37**, 911 (1988); P. Jung, P. Hanggi, and F. Marchesoni, Phys. Rev. A **40**, 5447 (1989).

[5] M. I. Dykman, Phys. Rev. A **42**, 2020 (1990).

[6] (a) A. Igarashi and T. Munakata, J. Phys. Soc. Jpn. **57**, 2439 (1988); (b) W. Ebeling and L. Schimansky-Geier, in *Noise in Nonlinear Dynamical Systems* (Ref. [1]), p. 279; (c) L. Schimansky-Geier and Ch. Zülicke, Z. Phys. B **79**, 451 (1990); (d) P. K. Koliopulos, S. R. Bishop, and G. D. Stefanou (to be published).

[7] P. V. E. McClintock and F. Moss, in *Noise in Nonlinear Dynamical Systems* (Ref. [1]), Vol. 3, p. 243.

[8] M. I. Dykman and M. A. Krivoglaz, Zh. Eksp. Teor. Fiz. **77**, 60 (1979) [Sov. Phys. JETP **50**, 30 (1979)]; in *Soviet Physics Reviews,* edited by I. M. Khalatnikov (Harwood Academic, New York, 1984), Vol. 5, p. 265.

[9] A. D. Ventzel and M. I. Freidlin, Russ. Math Surveys **25**, 1 (1970); M. I. Freidlin and A. D. Ventzel, *Random Perturbations in Dynamical Systems* (Springer-Verlag, Berlin, 1984).

[10] R. Phythian, J. Phys. A **10**, 777 (1977); L. Pesquera, M. A. Rodriguez, and E. Santos, Phys. Lett. **94A**, 287 (1983); P. Colet, H. S. Wio, and M. San Miguel, Phys. Rev. A **39**, 6094 (1989); H. S. Wio, P. Colet, M. San Miguel, L. Pesquera, and M. A. Rodriguez, Phys. Rev. A **40**, 7312 (1989).

[11] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).