Spin-Glass-Ferromagnetic-Paramagnetic Multicritical Point

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We study the criticality at the spin-glass-ferromagnetic-paramagnetic multicritical point in the $d=3$, $\pm J$ distribution, random-bond Ising model. Using high-temperature expansions to order T^{-34}, we estimate that the multicritical point N lies on the Nishimori line at $T_c/J=1.690\pm0.016$. Along this line the critical exponents are found to be $\gamma=1.80\pm0.15$ and $\nu=0.85\pm0.08$. The latter is clearly consistent with the rigorous exponent inequality $\nu\geq 2/d$. We also calculate the crossover exponent ϕ and show that the scaling axes at N are in agreement with the recent predictions of Le Doussal and Harris.

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An important issue in the study of randomly disordered magnetic systems is the competition between spin-glass and ferromagnetic order. While there are many experimental systems where these two types of order compete with each other leading to a complex phase diagram [1], the spin-glass-ferromagnetic-paramagnetic multicritical behavior has proven to be difficult to investigate. In theoretical studies of spin glasses such a multicritical point N takes on added significance following the work of Nishimori [2] who found that on a special line, in the parameter space of temperature and concentration of ferromagnetic bonds, many *exact* results can be obtained for the *random* spin systems. Nishimori used these results to put constraints on the topology of the spin-glass-ferromagnetic-paramagnetic phase boundaries. Following his work and some early numerical studies [3], it was suggested by Georges *et al.* that the multicritical point N must be located on this line, which they called the Nishimori line [4]. More recently, Le Doussal and Harris [5] and Le Doussal and Georges [6] have presented compelling renormalization-group and symmetry arguments for the multicritical point to lie on the Nishimori line and studied the multicritical behavior via $1/d$ and $6-\epsilon$ expansions.

While the existence of the multicritical point on the Nishimori line is now on firm theoretical grounds, our understanding of the multicritical behavior at N is far from satisfactory in the experimentally relevant case of three dimensions. Among the many studies for the $d=3$ case [3,7,8] the most extensive one is by Ozeki and Nishimori [8], who carried out Monte Carlo renormalization-group (MCRG) studies for the $\pm J$ model on large ($32\times32\times32$) lattices. For this model, the Nishimori line takes the form $2p-1=\tanh(J/kT)$, where p is the relative concentration of ferromagnetic bonds. They estimated that the critical point lies at $p_c=0.767\pm0.004$ ($T_c/J=1.68\pm0.025$). They also calculated the critical exponents along this line using the MCRG method. They obtained $\gamma/\nu=1.97\pm0.1$ and $\nu=0.51\pm0.06$. It was argued by Singh and Fisher [9] that these estimates were inconsistent with the rigorous exponent inequalities of Chayes *et al.* [10], which imply $\nu>2/d$. Ozeki and Nishimori [11] themselves later argued that their estimate of the exponent ν was in disagreement with the rigorous result of a nondivergent specific heat, if one assumed hyperscaling. Thus the study of the critical behavior along this line has remained in an unsatisfactory state. There has also not been a careful detailed study of the phase boundary in the vicinity of this critical point, and hence the multicritical behavior has not been elucidated numerically.

In this paper we develop high-order power-series expansions for various susceptibilities of interest in the variable $\tanh(J/kT)$ and study the multicritical behavior in some detail. We find that along the Nishimori line, $T_c/J=1.690\pm0.016$, which is in excellent agreement with the MCRG estimate of Ozeki and Nishimori. The critical exponents along the line are found to be $\gamma=1.80\pm0.15$, $\nu=0.85\pm0.08$, and $\eta=-0.10\pm0.02$. These are in clear disagreement with the MCRG estimates and are fully consistent with the exponent inequality of Chayes *et al.* It is shown that on one side of the Nishimori line the ferromagnetic susceptibility has the dominant singularity while on the other side it is the spin-glass susceptibility that has the stronger divergence. This implies that the multicritical point separating the paramagnetic, spin-glass, and ferromagnetic phases lies on this line. By studying appropriate derivatives of the susceptibility we calculate the crossover exponent ϕ and find that the scaling axes at the multicritical point are (i) along the Nishimori line and (ii) parallel to the temperature axis, in agreement with the predictions of Le Doussal and Harris.

We consider the nearest-neighbor Ising Hamiltonian on the simple cubic lattice, where the exchange constants $J_{i,j}$ are quenched random variables which take values $+J$ with probability p and $-J$ with probability $1-p$. For $p=1$ one obtains the pure Ising ferromagnet, whereas for $p=\frac{1}{2}$ one has the symmetric $\pm J$ Ising spin glass, which has been studied in great detail [12]. High-temperature expansions for this model, for general values of p, have been previously considered by several authors [7]. Following earlier work, two-variable expansions in powers of $v=\tanh(J/kT)$ and $2p-1$, complete to order v^{15}, were obtained for the simple cubic lattice by Singh and Fisher [13] and for d-dimensional hypercubic lattices by Fisch [14]. In this paper, by considering loci of the type

$$2p-1=\alpha v\,, \tag{1}$$

with α constant, we have been able to calculate exact high-temperature expansions to order v^{34}; this is the same order as previously obtained for the symmetric case [15]. Previous studies of the symmetric $\pm J$ model have demonstrated the need for long series in these random spin systems if one is to attain any reasonable assessment of the critical behavior [15,16].

We study the susceptibilities [16]

$$\chi_{m,n}=\frac{1}{N}\sum_{i,j}[\langle s_i s_j\rangle^m]^n\,, \tag{2}$$

where the angular brackets refer to thermal averaging and the square brackets to an ensemble average over the distribution of $J_{i,j}$. For $m=n=1$ we have the ferromagnetic susceptibility χ, for $m=2$, $n=1$ the spin-glass susceptibility $\chi_{\rm SG}$, and for $m=2$, $n=2$ an auxilliary susceptibility χ'. The expansions are obtained by the star-graph method [15]. The ensemble average for any bond leads to

$$[v_{i,j}^{2n}]=v^{2n},\quad [v_{i,j}^{2n+1}]=(2p-1)v^{2n+1}\,. \tag{3}$$

Along the loci of the type $2p-1=\alpha v$ the odd powers become αv^{2n+2}. Hence, along these loci every bond contributes only in even powers of v or J/T. This is the key feature that allows us to carry out the expansions to much higher orders [17].

Along the Nishimori line the series for the ferromagnetic and spin-glass susceptibilities become equal term by term. The expansions are [with $w=\tanh^2(J/kT)$]

$$\begin{aligned}\chi={}&1+6w+30w^2+150w^3+654w^4+2982w^5+11\,790w^6+50\,694w^7+186\,990w^8\\&+788\,038w^9+2\,730\,654w^{10}+11\,788\,806w^{11}+38\,258\,558w^{12}+177\,911\,046w^{13}\\&+523\,031\,214w^{14}+2\,741\,914\,182w^{15}+6\,592\,592\,526w^{16}+42\,207\,777\,222w^{17}+\cdots\,,\\ \chi'={}&1+6w^2+102w^4-96w^5+1998w^6-3792w^7+37\,878w^8-113\,040w^9\\&+789\,174w^{10}-2\,960\,928w^{11}+17\,558\,862w^{12}-74\,528\,976w^{13}+406\,171\,854w^{14}\\&-1\,856\,049\,840w^{15}+9\,672\,868\,326w^{16}-46\,152\,643\,776w^{17}+\cdots\,.\end{aligned}$$

For noninteger values of α the expansion coefficients are not integers. They are calculated on the computer in double precision [18].

We analyze the series using first-order inhomogeneous differential approximants [19]. We respect the function $f(w)$, whose power series in w has been calculated to a given order N, as a solution to a first-order inhomogeneous differential equation,

$$P_1^M(w)\frac{df}{dw}+P_2^L(w)f=P_3^J(w)+O(w^{M+L+J+2})\,. \tag{4}$$

Here P_1^M, P_2^L, and P_3^J are polynomials in w of order M, L, and J, respectively. The polynomials are determined by comparing coefficients of w^j with $0\le j\le M+L+J+1$ in (4). Arbitrariness of an overall multiplicative factor is removed by setting $P_1^M(0)=1$. The solutions to the differential equations have power-law singularities of the form $(w_c-w)^{-\gamma}$ at w_c given by $P_1^M(w_c)=0$, and γ given by $P_2^L/(dP_1^M/dw)$ evaluated at $w=w_c$.

We begin by analyzing the series along the Nishimori line ($\alpha=1$), where the ferromagnetic and spin-glass susceptibilities become equal term by term. For this series, as well as for the auxiliary susceptibility χ', we construct all approximants which use fourteen or more terms of the series with $M>4$, $L>2$, $J>1$, and $M\ge L$, $M\ge J$. This class of approximants, centered around the ones invariant [19] under Euler transformations (M, $L=M-2$, $J=M-2$), were found to be internally most consistent as found in the symmetric case [15]. In Fig. 1 we present histograms of estimated w_c for the two series. We estimate

$$w_c=0.282\pm0.004\,, \tag{5}$$

which leads to $T_c/J=1.690\pm0.016$. This is in excellent agreement with the results of Ozeki and Nishimori, who find $T_c/J=1.68\pm0.025$. Let γ and γ' be the critical exponents for the χ and χ' series, respectively. In Fig. 2 we present the estimates for these exponents, plotted against the estimated location of the critical point. We find that the exponent estimates are strongly correlated with the

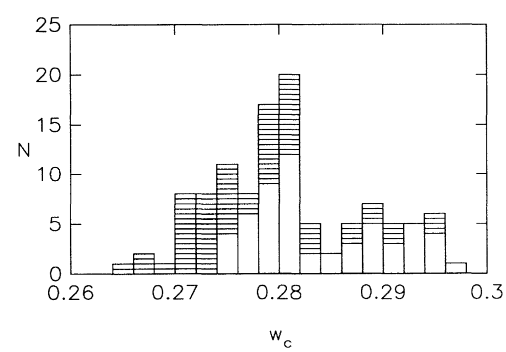

FIG. 1. Histogram of estimates of w_c along the Nishimori line. The shaded regions correspond to the χ' series and the unshaded regions to the χ series.

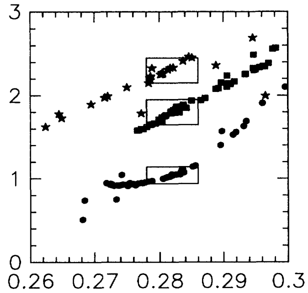

FIG. 2. Estimates of the exponent γ (represented by squares), γ' (represented by circles), and $\gamma+\phi$ (represented by stars) along the Nishimori line, as a function of the estimated w_c. The box shows the final accepted values based on taking $w_c=0.282\pm0.004$.

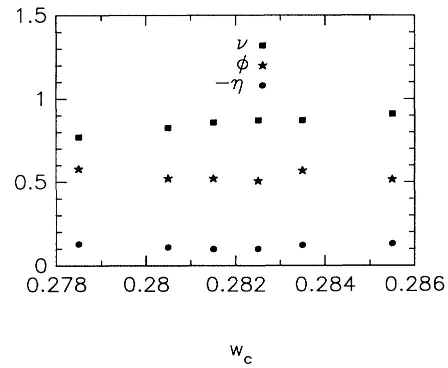

FIG. 3. Estimates of exponents ν, ϕ, and η as explained in the text as a function of the estimated critical point w_c.

location of the critical point as is invariably observed. Accepting the central region of the estimate for w_c in (5) we conclude that

$$\gamma=1.80\pm0.15,\quad \gamma'=1.04\pm0.10. \tag{6}$$

Assuming standard scaling, near the critical point, these exponents can be expressed in terms of η and ν as [16]

$$\gamma=(2-\eta)\nu,\quad \gamma'=(1-2\eta)\nu. \tag{7}$$

We note that the ratio of γ to γ' for approximants which give roughly the same w_c is nearly constant. Thus, to estimate ν and η we group all approximants with roughly the same w_c. The γ and γ' values for these approximants are averaged over, and η and ν are calculated. These estimates as a function of w_c are shown in Fig. 3. From there we obtain

$$\nu=0.85\pm0.08,\quad \eta=-0.10\pm0.02. \tag{8}$$

Thus we find, that while our ratio γ/ν is in agreement with that of Ozeki and Nishimori, ν itself is not. The estimates are clearly consistent with the rigorous bounds of Chayes *et al.* [10].

Let us now consider the case $\alpha\neq1$. The phase diagram is shown in Fig. 4. For $\alpha\gg1$, the two series appear to diverge along the same contour in the v-p plane with the ferromagnetic susceptibility having a stronger divergence. This is just as expected for a transition to a ferromagnetic state. As α approaches unity from above, the uncertainties in the series extrapolations increase. For α very close to unity, the two series appear to diverge with similar exponents, but with the ferromagnetic susceptibility diverging first (i.e., at a higher temperature). The critical exponents vary rapidly near the Nishimori point N, and should be regarded as effective exponents. We expect the true exponents to jump discontinuously from the random ferromagnetic value to the value at the multicritical point. On the other side of the Nishimori line ($\alpha<1$), a similar behavior appears with the spin-glass susceptibility diverging first, i.e., at a higher temperature. This is a clear indication that the phase boundary in Fig. 4 for $\alpha>1$ is a paramagnetic-ferromagnetic phase boundary, whereas that for $\alpha<1$ is a paramagnetic-spin-glass phase boundary, in agreement with the earlier suggestions [4]. Far from the Nishimori line, the paramagnetic-spin-glass phase boundary is obtained by monitoring the divergence of the spin-glass susceptibility. Here, as in the symmetric case [15], the uncertainties even in the location of the critical point are very large. The ferromagnetic susceptibility appears to be divergence free on this line, with possibly a weak nonanalyticity.

We now wish to obtain the scaling axes and the crossover exponent ϕ at the multicritical point N. Based on a symmetry argument Le Doussal and Harris have argued that the Nishimori line is one of the scaling axis. Their ϵ expansion around $d=6$ also suggested that the second scaling axis should be parallel to the temperature axis. In order to confirm this, we calculate the series expansions

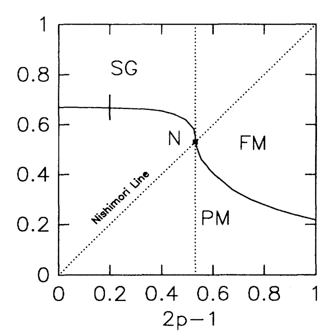

FIG. 4. Phase diagram for the $\pm J$ distribution, random-bond Ising model with p the relative concentration of ferromagnetic bonds and $v=\tanh(J/kT)$. The multicritical point N, and the two scaling axes, one of them being the Nishimori line, are shown by dotted lines. Typical uncertainty in the boundary between paramagnetic (PM) and spin-glass (SG) phases is shown by a vertical bar. The uncertainties in the paramagnetic-ferromagnetic (FM) phase boundary away from the Nishimori line are negligible on the scale of the diagram.

for $\chi_v = v\, \partial\chi_{SG}/\partial v$ along the Nishimori line. The series coefficients are

$$\chi_v = 12w + 120w^2 + 900w^3 + 4944w^4 + 27\,708w^5 + 119\,160w^6 + 592\,884w^7 + 2\,180\,832w^8 + 10\,781\,580w^9 + 34\,264\,536w^{10} + 190\,208\,964w^{11} + 506\,582\,880w^{12} + 3\,555\,480\,156w^{13} + 6\,896\,033\,208w^{14} + 68\,708\,311\,380w^{15} + 52\,261\,360\,896w^{16} + 1\,293\,272\,763\,660w^{17} + \cdots .$$

This series is analyzed by the same method as the others. The estimated exponent values are also shown in Fig. 2. We obtain

$$\gamma + \phi = 2.3 \pm 0.15 . \tag{9}$$

The absence of the $\gamma+1$ divergence clearly shows that one of the scaling axes is given by $p = p_c$, i.e., it is parallel to the temperature axis. In other words, the multicritical behavior has the scaling form

$$\chi_{SG} = (p - p_c)^{-\gamma} F\left[\frac{v - 2p + 1}{(p - p_c)^{\phi}}\right], \tag{10}$$

where $F(x)$ is the scaling function. This also implies that the slope of the phase boundary dv_c/dp diverges at the multicritical point. Finally, by using the same method as that used for ν and η, we calculate the crossover exponent to be

$$\phi = 0.54 \pm 0.04 . \tag{11}$$

We note that the susceptibility exponent on approaching the multicritical point parallel to the temperature axis is given by γ/ϕ.

In summary, in this paper we have investigated the phase diagram and the critical behavior in an asymmetric Ising spin-glass model via high-temperature expansions. Clear evidence has been presented that spin-glass and ferromagnetic order interchange dominance on the Nishimori line, confirming earlier conjectures that the multicritical point N separating ferromagnetic, spin-glass, and paramagnetic phases must lie on it. The multicritical behavior at N including the scaling axes and the critical exponents have been estimated. We find that the exponent ν along the Nishimori line differs significantly from those obtained earlier by Ozeki and Nishimori, and is fully consistent with the bounds of Chayes *et al.* [10]. We hope our work would stimulate further experimental interest in studying this multicritical point.

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