Chaotic Properties of the Interacting-Boson Model: A Discovery of a New Regular Region

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The chaotic properties of the interacting-boson model of nuclei are studied both classically and quantum mechanically. Classical phase diagrams are constructed in the general parameter space of the Hamiltonian known as Casten's triangle. They lead to a discovery of a nearly regular region which is probably related to an unknown approximate symmetry. Analysis of the quantal fluctuations of the spectrum and of the electromagnetic $B(E2)$ intensities confirms the classical results. The phase diagrams may be useful in identifying "chaotic" nuclei within the nuclear periodic table.

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The interacting-boson model (IBM) [1,2] of heavy nuclei has been successful in describing phenomenologically the low-lying energy levels and electromagnetic transition intensities of a large number of nuclei [3]. As an algebraic model with a finite Hilbert space it is relatively easy to solve. In particular, when a dynamical symmetry occurs it has a closed-form solution. The model is known to have three dynamical symmetry limits which correspond to vibrational, rotational, and γ-unstable nuclei. A system which possesses a dynamical symmetry is also completely integrable, i.e., has a complete set of constants of the motion in involution. When the dynamical symmetry is broken the system may become nonintegrable. The long-time dynamics of nonintegrable systems in few degrees of freedom has recently been a subject of numerous investigations. Of particular interest is the onset of classical chaotic motion and its signatures in the associated quantal system [4–6]. We have recently studied the transition between the rotational and γ-unstable limits of the interacting-boson model. While the system at these limits is regular, as expected, it becomes mostly chaotic in between. This was shown both classically [7,8] and quantum mechanically [7] where statistical fluctuations of levels and $E2$ intensities are described by the Gaussian orthogonal ensemble (GOE) of random matrices [9].

To study the onset of chaos in the low-lying collective part of the nuclear spectrum it is important to investigate the dynamics of the general interacting-boson Hamiltonian. The purpose of this Letter is to describe these new realistic calculations and in particular to present a strikingly surprising result: the discovery of a new family of IBM Hamiltonians which is characterized by almost regular dynamics and which does not belong to any of the known dynamical symmetry limits. This family probably possesses an unknown approximate symmetry which leads to its regular dynamical behavior. Our result is especially surprising in that it went unnoticed in a model which has been studied extensively for more than a decade. We also present for the first time classical phase diagrams which describe the degree of chaos in the complete parameter space of the IBM Hamiltonian. Our quantal studies which include analysis of both the level statistics and the $B(E2)$ intensity fluctuations are in accord with the classical results.

The general IBM Hamiltonian is constructed through the generators of a U(6) algebra and is most economically parametrized in the self-consistent Q formalism [10]:

$$H=E_0+c_0\hat{n}_d+c_2Q^\chi\cdot Q^\chi+c_1\mathbf{L}^2\,. \tag{1}$$

Here $\hat{n}_d=d^\dagger\cdot\tilde{d}$ is the number of d bosons, $\mathbf{L}$ is the angular momentum, and Q^χ is a quadrupole operator,

$$Q^\chi=(d^\dagger\times\tilde{s}+s^\dagger\times\tilde{d})^{(2)}+\chi(d^\dagger\times\tilde{d})^{(2)}\,, \tag{2}$$

which depends on a parameter χ. The Hamiltonian (1) has three dynamical symmetries for which it can be expressed in terms of the Casimir invariants of a chain of subalgebras of U(6),

$$\mathrm{U}(6)\supset\begin{Bmatrix}\mathrm{U}(5)\supset\mathrm{O}(5)\\ \mathrm{SU}(3)\\ \mathrm{O}(6)\supset\mathrm{O}(5)\end{Bmatrix}\supset\mathrm{O}(3)\quad\begin{matrix}(\mathrm{I})\\ (\mathrm{II})\\ (\mathrm{III})\end{matrix}\,. \tag{3}$$

Chain (I) corresponds to $c_2=0$ (vibrational nuclei), chain (II) is obtained when $c_0=0$ and $\chi=-\sqrt{7}/2$ (rotational nuclei), and chain (III) is described by $c_0=0$ and $\chi=0$ (γ-unstable nuclei).

A semiclassical (mean-field) description in which the inverse boson number $1/N$ plays the role of $\hbar$ is obtained [11] by using coherent states [12] $|\boldsymbol{\alpha}\rangle\equiv|\alpha_s,\alpha_{-2},\ldots,\alpha_2\rangle$ as discussed in Refs. [7] and [8]. To take the classical limit $N\to\infty$ we first replace $\boldsymbol{\alpha}$ by $\tilde{\boldsymbol{\alpha}}=\boldsymbol{\alpha}/\sqrt{N}$. Renaming $\tilde{\alpha}$ by $\boldsymbol{\alpha}$, the constraint on the total boson number becomes N independent, $\boldsymbol{\alpha}^2=1$. $\boldsymbol{\alpha}$ and $i\boldsymbol{\alpha}^*$ play the role of canonical conjugate variables for the classical Hamiltonian $h(\alpha,\alpha^*)\equiv\langle\boldsymbol{\alpha}|H|\boldsymbol{\alpha}\rangle/N$, where

$$h(\alpha,\alpha^*)=\epsilon_0+\bar{c}[\eta n_d-(1-\eta)q^\chi\cdot q^\chi]+\bar{c}_1 l^2\,. \tag{4}$$

Here n_d, q^χ, and l are c-functions obtained from the operators $\hat{n}_d$, Q^χ, and $\mathbf{L}$, respectively, through the replacement $s^\dagger,d_\mu^\dagger\to\alpha_s^*,\alpha_\mu^*$ and $s,d\to\alpha_s,\alpha_\mu$. $\boldsymbol{l}$ is then the angular momentum per boson. The parameters of the classical Hamiltonian (4) are related to those of the quantal Hamiltonian (1) through

$$\epsilon_0=\frac{E_0}{N},\quad \bar{c}_1=Nc_1,$$
$$\frac{\eta}{1-\eta}=-\frac{c_0}{Nc_2}\quad(0<\eta<1),\quad \bar{c}=\frac{c_0}{\eta}. \tag{5}$$

Notice that relations (5) depend on the boson number N, so that for fixed values of the parameters of the classical Hamiltonian the corresponding quantal parameters depend on N.

For a given angular momentum $l^2=\text{const}$, $\bar{c}_1$ does not affect the character of the dynamics and we may choose $\bar{c}_1=0$ and $\epsilon_0=0$ with no loss of generality. By proper rescaling of the Hamiltonian h we may then take $\bar{c}=1$. It follows that the only relevant parameters are χ and η, where $-\sqrt{7}/2\le\chi\le 0$ and $0\le\eta\le 1$. The U(5) limit is then obtained for $\eta=1$, the SU(3) limit for $\eta=0$, $\chi=-\sqrt{7}/2$, and the O(6) limit for $\eta=0$, $\chi=0$. The parameter space can be described by a triangle whose base describes the χ axis and its height, the η axis, as in Fig. 1.

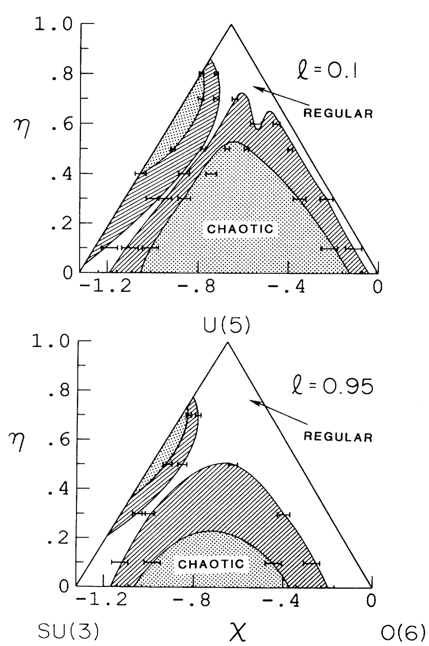

FIG. 1. Classical phase diagrams in the η-χ plane for a given l (spin per boson). Top: $l=0.1$; bottom: $l=0.95$. A point (η,χ) inside the triangle corresponds to a general IBM Hamiltonian of the form (4). The χ value of a point inside the triangle is given by the intersection with the $\eta=0$ line of a straight line drawn from the upper vertex of the triangle through the given point. The solid lines inside the triangle separate regular (where the fraction of chaotic volume σ is less than 0.3) and chaotic ($\sigma>0.7$; dotted) regions. The hatched area is the transition region ($0.3<\sigma<0.7$). Notice in particular the regular strip inside the triangle that connects the SU(3) and U(5) limits. Also, at higher spins the system becomes less chaotic.

This is a rotated version of Casten's triangle [13]. To read the χ coordinate for a point inside the triangle we take a straight line from the upper vertex through the given point and read its intersection with the $\eta=0$ line. The three vertices of the triangle correspond to the three dynamical symmetries.

To investigate the classical dynamics we have used Monte Carlo techniques to calculate the fraction of chaotic volume σ on a given energy-angular-momentum surface. A chaotic trajectory is characterized by a positive maximal Lyapunov exponent. Figure 1 shows a "dynamical" phase diagram in the χ-η variables for fixed values of the classical angular momentum. Here σ is averaged over the energy. The regular region corresponds to $\sigma<0.3$ and the chaotic (dotted) areas to $\sigma>0.7$. The transition regions $0.3<\sigma<0.7$ are the hatched areas. Our previous studies [7,8] were restricted to the base of the triangle ($\eta=0$), where we have observed a chaotic behavior for intermediate χ's in the transition between the SU(3) and O(6) limit. A similar behavior is observed in Fig. 1 along the transition line between SU(3) and U(5). The regular strip along the transition between O(6) and U(5) indicates that there is no onset of chaos there. This is expected since along that line O(5) is a common subalgebra and the two Casimir invariants of O(5) are constants of the motion. Together with N, $\mathbf{L}^2$, L_z, and H they form a complete set of constants in involution.

A surprising result is the narrow regular strip which connects the SU(3) and U(5) vertices but is inside the triangle. It exists in the same region of the parameter space for any angular momentum. It is important to note that strictly speaking the system is nonintegrable along this strip since there are some chaotic trajectories. However, their fraction is small ($\sigma<0.3$) and their Lyapunov exponents are relatively small. The system is thus nearly regular which suggests an approximate symmetry in this region. This is a very interesting property of the family of IBM Hamiltonians which was never noticed before.

The behavior of the dynamics as a function of spin can also be deduced from Fig. 1. It is clearly seen that the chaotic regions shrink at high spin. The study of the classical motion described by (4) allows us to find the dependence of chaos on the excitation energy of the nucleus. Figure 2 shows phase diagrams in the χ-ϵ plane (where ϵ is the excitation energy per boson) at a given angular momentum l and which correspond to several values of η: 0, 0.5, and 0.7. These correspond to three cuts parallel to the basis in the triangle diagram. The regular, chaotic, and transition regions are defined similarly to Fig. 1. From the $\eta=0$ diagram we see that the generic situation for an intermediate χ is that at low energies the motion is regular but it gradually becomes chaotic at higher excitation energies. When $\eta>0$ this singly connected chaotic region breaks into two parts. The narrow transition region which separates them corresponds to points in the

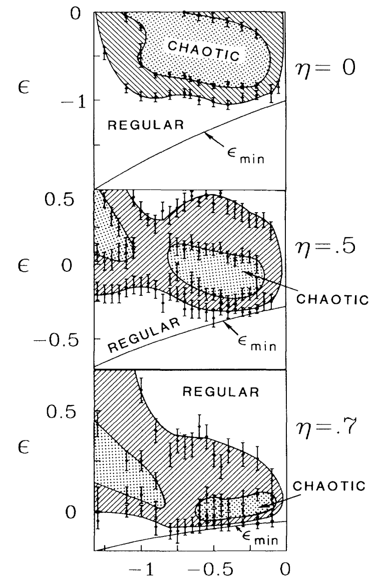

FIG. 2. Classical phase diagrams at constant $l=0.1$ in the ϵ (energy per boson)–χ plane for $\eta=0$ (top), $\eta=0.5$ (middle), and $\eta=0.7$ (bottom). Regular and chaotic regions are denoted as in Fig. 1. $\epsilon_{\min}$ denotes the lowest possible energy for a given χ. Notice that the chaotic region for $\eta=0$ breaks into two disconnected regions for $\eta=0.5$ and 0.7. This is due to the regular strip of Fig. 1.

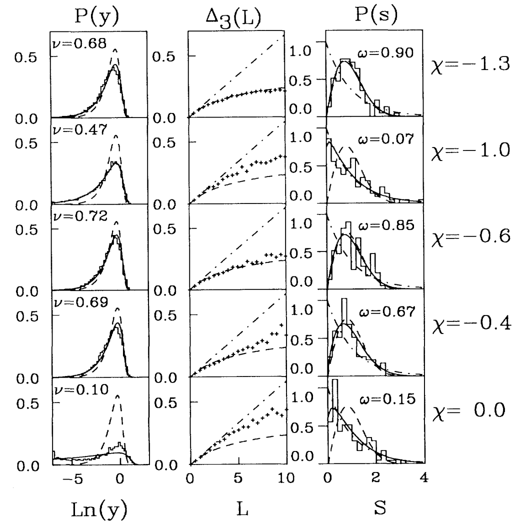

FIG. 3. Statistical fluctuations of the spectrum and the electromagnetic transitions in the quantal IBM Hamiltonian (1) with $N=25$ bosons for $\eta=0.5$ [see Eq. (5)] and several values of χ. Shown are the level spacing distribution $P(S)$ (histograms in the right column), the Δ_3 statistics (plusses in the middle column), and the $B(E2)$ distribution $P(y)$ (histograms in the left column) for the $J^\pi=2^+$ states. The dashed lines are the GOE limit and the dot-dashed lines are the Poisson limit. The classical transition of chaotic→ regular→ chaotic→ regular observed in Fig. 1 as χ changes between -1.3 and 0 across the $\eta=0.5$ line is reproduced in the quantal case.

regular strip of Fig. 1. Thus, this newly discovered almost regular region does not become strongly chaotic at any energy. Another interesting phenomenon is observed in regions within the triangle to the immediate right of the regular strip: As a function of increasing energy we first observe a rapid transition from regular to chaotic behavior but then a much more gradual transition to a more regular behavior at high energies.

Does the above regular strip show up in the quantal properties of the model? To answer that we have studied the corresponding quantal systems where the connection between the parameters of the classical and quantal system is given by Eqs. (5). In the quantal model we have investigated the statistical fluctuations of the spectrum and of the $B(E2)$ intensities. For the spectrum we use two standard measures [4–6]: the level spacing distribution $P(S)$ and the Dyson-Metha statistics $\Delta_3(L)$. For the $E2$ $J^\pi \to J^\pi$ transitions we have analyzed the distribution of intensities [14] $P(y)$, where $y \equiv B(E2;i\to f)$. See Ref. [7] for details. Figure 3 shows the quantal results for the $J^\pi=2^+$ states. To obtain good statistics we have chosen a relatively large number of bosons $N=25$. We emphasize, however, that similar effects are seen for more realistic values of N. We chose a cut $\eta=0.5$ parallel to the triangle's base. We remark that $J^\pi=2^+$ for 25 bosons corresponds to $l=0.08$ (which is close to the $l=0.1$ case of Fig. 1). According to Fig. 1 the classical system makes the succession of transitions chaotic→ regular → chaotic→ regular as χ is increased from $-\sqrt{7}/2$ to 0. The respective values of χ are shown in Fig. 3 from top to bottom. The dashed lines in this figure describe the completely chaotic limit (GOE) while the dot-dashed lines describe the integrable limit (Poisson). In the column describing $P(S)$, ω is the parameter of the Brody distribution [9] ($\omega=1$ for GOE and $\omega=0$ for Poisson). ν in the $P(y)$ column is from the fit with a χ^2 distribution in ν degrees of freedom [15] ($\nu=1$ for GOE and ν decreases towards 0 for a regular system [14]),

$$P_\nu(y) = Ay^{\nu/2-1}\exp(-\nu y/2\langle y\rangle)\,. \quad (6)$$

All the above three quantal measures are found to be consistent with the classical results: chaotic for $\chi=-1.3$, regular for $\chi=-1$, chaotic again for $\chi=-0.6$, and then intermediate for $\chi=-0.3$ and regular for $\chi=0$. Notice in particular the regular character of the quantal fluctuations for $\chi=-1$. This point is in the newly discovered regular strip of Fig. 1.

In conclusion, we have studied classical and quantum chaos in the general interacting-boson model of nuclei. The most striking result is the discovery of a nearly regular region inside Casten's triangle which is not related to any of the three known dynamical symmetries of the model. Since the model is realistic for a large volume of nuclei, our results may be useful in classifying the degree of chaoticity in the low-lying collective states of nuclei within the nuclear periodic table.

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