Path Integrals as Discrete Sums

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We present a new formulation of Feynman's path integral, based on Voronin's theorems on the universality of the Riemann zeta function. The result is a discrete sum over "paths," each given by a zeta function. A new measure which leads to the correct quantum mechanics is explicitly given.

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Even though the "path integral" plays a fundamental role in quantum theories, our ability to compute with it or to use it formally for establishing general results is rather limited. For actual computations on the lattice one is essentially restricted to the Monte Carlo method which has its own limitations. Formally, one uses the functional formalism as a generator of perturbation theory, and the existence of the continuum limit is often hard to establish. Guided by Wheeler's intuitive idea of quantum theory as an "average over histories" we propose a new formulation of the path integral based on a countable set of "paths" which are in some sense "quantum-mechanically complete." Such a formulation will have several advantages. The most important is the possible replacement of multidimensional (or even infinite-dimensional functional integrals) with weighted one-dimensional integrals, and eventually we hope to avoid the need for introducing a lattice. In addition, the fact that each "path" will be defined by an analytic function with well-known properties and integral representations will make formal manipulations possible where they have not been with the standard formulation. From the computational point of view one can get an alternative to the Monte Carlo method.

In a recent paper [1] we introduced a new definition of the Feynman path integral and expressed it as a discrete sum over "paths." Each path is given by a vector whose components are zeta functions evaluated at points in the critical region. We also showed how a new measure (or Jacobian) can be determined to give the correct quantum mechanics. We checked our results by carrying out extensive numerical calculations on the Connection Machine to test our new method. The agreement was extremely encouraging.

In this Letter we give a brief summary of our results, which have been improved by defining the paths in terms of $\ln|\zeta|$, a modification which allows us to give an explicit expression for the measure. As in Ref. [1], we discuss only Euclidean quantum mechanics, and the generalization to quantum fields will be given elsewhere.

Euclidean quantum mechanics, with Euclidean time x, $0 \le x \le L$, is described by a field $\phi(x)$. To define the path integral one introduces a lattice on the interval $0 \le x \le L$, with lattice spacing a, $x_j = ja$, $j = 1, \dots, \nu$, and $\nu \equiv L/a$. The partition function Ω is given by

$$\Omega(\nu) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left[\prod_{j=1}^{\nu} d\phi(x_j)\right] e^{-S(\phi(x_1),\dots,\phi(x_\nu))} \tag{1}$$

and

$$S(\phi) = \tfrac{1}{2}\sum_{j=1}^{\nu} \frac{[\phi(x_j)-\phi(x_{j+1})]^2}{a} + \tfrac{1}{2} m^2 a \sum_{j=1}^{\nu} \phi^2(x_j) + a\sum_{j=1}^{\nu} V(\phi(x_j)) . \tag{2}$$

The integral in Eq. (1) is an integral over R^ν. However, given the positivity of S, and the fact that $V(\phi)$ must be bounded from below, one can well approximate the integral by integrating over a finite volume $V \in R^\nu$.

Our main idea is to use the following version of Voronin's [2] theorem. We start with a fixed set of ν complex numbers, $s_1, s_2, \dots, s_\nu$, where $s_j \neq s_k$ if $j \neq k$, and $\frac{1}{2} < \mathrm{Re}\, s_j < 1$, for $j = 1, \dots, \nu$. Then we define the set of vectors $\mathbf{u}(n)$, $\mathbf{u}(n) \in R^\nu$, and $n = 1, 2, \dots$, as

$$\mathbf{u}(n) \equiv \{\ln|\zeta(s_1 + in\Delta)|, \dots, \ln|\zeta(s_\nu + in\Delta)|\} , \tag{3}$$

where $\Delta > 0$, arbitrary, and fixed, and ζ is the standard Riemann zeta function. Voronin's theorem asserts that the set $\mathbf{u}(n)$, $n \in Z$, is dense in R^ν. We also have an important result about the density of the vectors $\mathbf{u}(n)$ in R^ν when $n = 1, 2, \dots, N$ and N is large. Let $\mathcal{L}(n;N)$ be the set of integers defined by

$$\mathcal{L}(n;N) = \left\{ l \in [1, \dots, N] \,\Big|\, |\mathbf{u}(n) - \mathbf{u}(l)| < \epsilon \right\} . \tag{4}$$

The remarkable fact is that as $N \to \infty$, $\mathcal{L}(n;N)$ is not only nonempty but has a positive density $\rho_\nu(n)$ defined as

$$\rho_\nu(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} \chi_{\mathcal{L}(n;N)}(l) , \tag{5}$$

where $\chi_{\mathcal{L}}$ is the characteristic function of $\mathcal{L}$.

To apply Voronin's theorem to a particular field configuration, i.e., path, $\phi(x_1), \dots, \phi(x_\nu)$, we first introduce a linear mapping between the lattice points, $x_1, \dots, x_\nu$, in space-time and the points $s_1, \dots, s_\nu$ in the critical region. We choose all the s_j's to lie on a vertical line, i.e., $\mathrm{Re}\, s_j = \sigma$, $\frac{1}{2} < \sigma < 1$, $j = 1, \ldots, \nu$, and keep σ fixed. The advantages of this last choice will become apparent later. Our mapping is

$$x_j \leftrightarrow (\sigma + ijh), \quad j = 1, \ldots, \nu, \tag{6}$$

where $h > 1$ and fixed. We define our nth path by the configuration

$$\phi^{(n)}(x_j) \equiv \ln|\zeta(\sigma + ijh + in\Delta)|, \quad j = 1, \ldots, \nu. \tag{7}$$

Here to separate our paths we take $\Delta \gg h$, and $\Delta > h\nu$.

As n ranges from $n = 1$ to $n = N \to \infty$, it is clear that for any field configuration $\Phi \in R^\nu$, $\Phi = \{\tilde{\phi}(x_1), \ldots, \tilde{\phi}(x_\nu)\}$, we can find a large set of positive integers, $\mathcal{L}(\Phi)$, such that

$$|\tilde{\phi}(x_j) - \ln|\zeta(\sigma + ijh + il\Delta)|| < \epsilon, \quad j = 1, \ldots, \nu, \tag{8}$$

for all $l \in \mathcal{L}(\Phi)$.

It now follows from the standard definition of the integral (1) that one can write

$$\Omega(\nu) = \sum_{n=N_0}^{N} \frac{e^{-S(n;\nu)}}{\rho_\nu(n)} + O(N^{-1/\nu}). \tag{9}$$

Here,

$$S(n;\nu) = \sum_{j=1}^{\nu} \left[\frac{[\gamma_\sigma(j;n) - \gamma_\sigma(j+1;n)]^2}{2a} + \tfrac{1}{2} m^2 a \gamma_\sigma^2(j;n) + aV(\gamma_\sigma(j;n)) \right] \tag{10}$$

and

$$\gamma_\sigma(j;n) = \ln|\zeta(\sigma + ijh + in\Delta)|. \tag{11}$$

We must take N large, and $N \gg N_0$. By summing over n we sum over all paths, but the density $\rho_\nu(n)$ insures that we have the correct Jacobian for quantum mechanics. For an estimate of the error in (9) one should check Ref. [1] for more details.

The main question in this paper is to find the expression for $\rho_\nu(n)$, the density of the vectors $\mathbf{u}(l)$ in the neighborhood of a given $\mathbf{u}(n)$.

We first discuss the case $\nu = 1$. Here $\rho_1(n)$ can be computed once we know the asymptotic probability for having $|\zeta(\sigma + it)|$, for a randomly chosen $t \gg 1$, be such that $r_1 < |\zeta(\sigma + it)| < r_2$. We call this probability density $\mathcal{P}_\sigma(r)$, and write

$$\mathrm{Prob}(r_1 < |\zeta(\sigma + it)| < r_2) \equiv \int_{r_1}^{r_2} \mathcal{P}_\sigma(r)\, dr, \quad \tfrac{1}{2} < \sigma < 1. \tag{12}$$

In the notation of Ref. [1], $\mathcal{P}_\sigma(r) \equiv \bar{P}_\sigma(r) r$. The moments of $\mathcal{P}_\sigma(r)$ are known for $0 \le \mathrm{Re}\, k \le 2$,

$$\int_0^\infty \mathcal{P}_\sigma(r) r^{2k}\, dr \equiv \lim_{T\to\infty} \frac{1}{T} \int_1^T |\zeta(\sigma + it)|^2\, dt = F_k(2\sigma) \tag{13}$$

and

$$F_k(2\sigma) = \prod_p {}_2F_1\left(k, k; 1; \frac{1}{p^{2\sigma}}\right), \tag{14}$$

where ${}_2F_1(a,b;c;z)$ is the standard hypergeometric function, and $\prod_p$ is a product over all primes. These results are in Ref. [3] (see Theorem 7.11 and also Sec. 7.19). We can obtain an explicit expression for $\mathcal{P}_\sigma(r)$ by taking the inverse Mellin transform and get

$$\mathcal{P}_\sigma(r) = \frac{r^{-c}}{2\pi} \int_{-\infty}^{+\infty} d\lambda\, (r)^{-i\lambda} F_{i\lambda/2 + (c-1)/2}(2\sigma), \tag{15}$$

where we can take any real c such that $1 < c < 5$. As shown in Ref. [1], the integral in (15) is absolutely convergent. From Eq. (15) it is now easy to obtain the distribution function of the values of $\ln|\zeta|$, which we call $W_\sigma(\gamma)$,

$$\mathrm{Prob}(\gamma_1 < \ln|\zeta(\sigma + it)| < \gamma_2) \equiv \int_{\gamma_1}^{\gamma_2} W_\sigma(\gamma)\, d\gamma$$

for a randomly chosen $t \gg 1$. It is clear that $W_\sigma(\gamma) \equiv \mathcal{P}_\sigma(e^\gamma) e^\gamma$, and hence

$$W_\sigma(\gamma) = e^{-(c-1)\gamma} \int_{-\infty}^{+\infty} d\lambda\, e^{-i\lambda\gamma} F_{i\lambda/2 + (c-1)/2}(2\sigma) (2\pi)^{-1} \tag{16}$$

with $1 < c < 5$. The probability density $W_\sigma(\gamma)$ is an asymptotic distribution in the sense that if we take an interval $T_0 < t < T$, $T \gg T_0$, and compute a large ensemble of $\ln|\zeta(\sigma + it_j)|$ values, then the resulting histogram for the distribution of values for this ensemble will approach $W_\sigma(\gamma)$ as $T \to \infty$, and T_0 is kept fixed. However, it is fortunate that even for an interval $T_0 = O(10^6)$ and $T = O(10^9)$ the computed histogram and the exact result for $W_\sigma(\gamma)$ are quite close to each other. This fact is shown in Fig. 1. The region $T_0 < t < 10^9$ is easily acces-

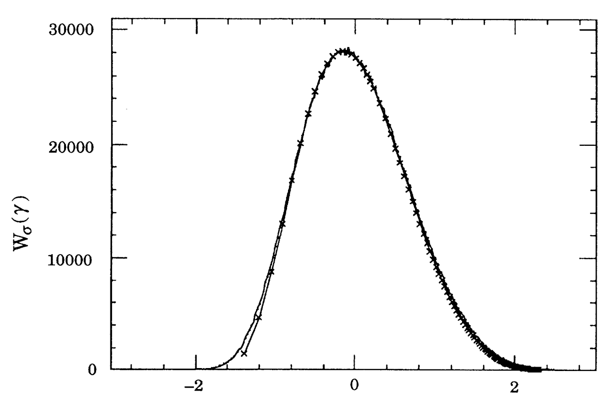

FIG. 1. Distribution of the values of $\gamma = \ln|\zeta|$ at $\sigma = 0.75$. The crosses denote points computed from the exact formula (16), and the line is the histogram from computing a large sample.

sible to compute.

The moments of $W_\sigma(\gamma)$ can be given exactly. It follows from Eqs. (13) and (16) that

$$\langle[\ln|\zeta(\sigma+it)|]^l\rangle=\lim_{T\to\infty}\frac{1}{T}\int_1^T[\ln|\zeta(\sigma+it)|]^l dt\equiv\int_{-\infty}^{+\infty}W_\sigma(\gamma)\gamma^l d\gamma=\left(\frac{1}{2}\frac{d}{dk}\right)^l F_k(2\sigma)\bigg|_{k=0},\quad \tfrac{1}{2}<\sigma<1,\tag{17}$$

with l a positive integer. For $l=1$ the result is zero. The result for $l=2$ can also be explicitly calculated and one gets

$$\langle[\ln|\zeta(\sigma+it)|]^2\rangle=-\tfrac{1}{2}\sum_p L(p^{-2\sigma}),$$

where $L(z)$ is the Spence function, $L(z)=\int_0^z \ln(1-x)\times dx/x$.

For $\nu=1$, the density $\rho_1(n)$ is simply given by $\rho_1(n)=W_\sigma(\gamma(1;n))$, with $\gamma(1;n)$ defined in Eq. (11). For $\nu>1$, we need the probability for obtaining a vector $\mathbf{u}(n)\in R^\nu$. With h fixed and $h>1$, the values of $\gamma_\sigma(j;n)$ and $\gamma_\sigma(j+l;n)$ are uncorrelated. This was explicitly tested in Ref. [1]. Thus the probability density for a specific $\mathbf{u}(n)$ is the product of the independent probabilities for each component. This leads to a factorization of $\rho_\nu(n)$ and we get

$$\rho_\nu(n)=\prod_{j=1}^{\nu}W_\sigma(\gamma_\sigma(j;n))\tag{18}$$

with $\gamma(j;n)$ defined in Eq. (11).

We now have the following explicit formulas for both the partition function and the Green's function in Euclidean quantum mechanics:

$$\Omega(\nu)=\sum_{n=1}^{N}e^{-S(n;\nu)}\left(\prod_{j=1}^{\nu}W_\sigma(\gamma_\sigma(j;n))\right)^{-1}+O(N^{-1/\nu})\tag{19}$$

and

$$G(j,l)=[\Omega(\nu)]^{-1}\sum_{n=1}^{N}\gamma_\sigma(j;n)\gamma_\sigma(l;n)e^{-S(n;\nu)}\left(\prod_{j=1}^{\nu}W_\sigma(\gamma_\sigma(j;n))\right)^{-1}+O(N^{-1/\nu}).\tag{20}$$

In Eqs. (19) and (20), $W_\sigma(\gamma)$ is explicitly given by Eq. (16), $\gamma_\sigma(j;n)$ is defined in Eq. (11), and the action $S(n;\nu)$ is given in Eq. (10).

In Ref. [1] extensive numerical calculations were carried out to check the validity of Eqs. (19) and (20). We applied the method to both the harmonic and anharmonic oscillators, and compared our results with older methods and with exact results. This was even done for the propagator of a free theory where the lattice result is also exactly known. In addition, we were able to calculate the ground-state energy of the harmonic oscillator with good accuracy. This involves calculating $\ln\Omega$, and computing the partition function is much more difficult than computing averages of observables in any method. A very encouraging feature of our calculations is the fact that we obtain good results with N, the number of paths, being such that $N\ll e^\nu$. This is a strong indication that there exists a manageable subset of paths that dominate the physics. But the greatest potential interest in our method is the possibility of a direct continuum formation which we discuss in more detail in (c) below.

We close this paper with a few remarks.

(a) The formula (19) can be generalized to scalar quantum field theories in higher dimensions, $d=2,3,4$.

(b) As mentioned in the introduction, our motivation for exploring this new method has both formal and computational aspects. The formal consequences of our results are being pursued, and we hope to report on them in the future. For the computational uses of our formula one needs an algorithm to select a subset of integers n which dominate the sum over paths. This will lead to an alternative to the Monte Carlo method which will allow us to do what was difficult before. For example, one could do ϕ^4 field theory with complex coupling.

(c) The continuous form of Voronin's theorem, i.e., Theorem 1 of Ref. [1], leads us to contemplate a far reaching conjecture. Namely, this concerns taking the limit $a\to 0$, where a is the lattice spacing. If in this limit a measure, $\rho_\infty(n)$, exists, then essentially any quantum-mechanical problem can be reduced to quadratures.

For $0\le x\le L$, we then can use as a "quantum-mechanically complete" set of fields, $\phi_\sigma^{(n)}(x)$, the following:

$$\phi_\sigma^{(n)}\equiv\gamma_\sigma(x;n)\equiv\ln|\zeta(\sigma+ix\Delta/L+in\Delta)|,\tag{21}$$

where any fixed $\Delta>0$ will do, and $\frac{1}{2}<\sigma<1$.

The Green's function for any action can now be written as a series:

$$G(x,y)=\lim_{N\to\infty}\left\{\left[\sum_{n=N_0}^{N}\gamma_\sigma(x;n)\gamma_\sigma(y;n)\alpha_n\right]\Big/\left[\sum_{n=N_0}^{N}\alpha_n\right]\right\}.\tag{22}$$

Given $\rho_\infty(n)$ all the positive coefficients α_n are known explicitly,

$$\alpha_n\equiv[\rho_\infty(n)]^{-1}e^{-S(n)}$$

and

$$S(n)\equiv\int_0^L dx\left[\frac{1}{2}\left(\frac{\partial\gamma_\sigma(x;n)}{\partial x}\right)^2+\tfrac{1}{2}m^2\gamma_\sigma^2(x;n)+V(\gamma_\sigma(x;n))\right].\tag{23}$$

The important thing to remember is that $\rho_\infty(n)$ depends only on the properties of the Riemann zeta function and has no physics in it. All the physics enters through $S(n)$. Clearly, the existence of $\rho_\infty(n)$ and an explicit formula for it would be a remarkable achievement. It will make the introduction of a lattice unnecessary.

(d) From Fig. 1 one can see that both the exact $W_\sigma(\gamma)$ and the histogram computed for it look roughly like shifted Gaussians. The maximum is not at $\gamma=0$. Selberg [4] has shown that distribution of values for $\ln\zeta(\frac{1}{2}+it)/\sqrt{2\pi\ln\ln t}$ is actually a Gaussian. However, computationally this does not become apparent except for much higher values of t than we have used. For the physics applications we envisage, we prefer to work with values of $\sigma>\frac{1}{2}$, where, as seen in Fig. 1, the asymptotic distribution is reached even for $10^6<t<10^9$. Also the factor $\sqrt{\ln\ln t}$ introduces additional complications. At this stage we are satisfied with a $W_\sigma(\gamma)$ which is given explicitly and which can be easily computed.

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