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New Hierarchy of Colored-Braid-Group Representations

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Novel solutions of the Yang-Baxter relation for a series of multistate vertex models are presented in the braid-group limit. The obtained family of solutions gives a sequence of "colored"-braid-group representations. Possible generalizations of the multivariable Alexander polynomial are discussed.

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Recently, a close connection between the theory of integrable systems and the knot theory was found [1,2]. The key is the Yang-Baxter relation [3] which originally was the applicability condition of the Bethe-ansatz technique for one-dimensional many-body problems.

In the celebrated work of Jones [1], an explicit braid-group representation derived from the von Neumann algebra theory was utilized to construct a new link polynomial called the Jones polynomial. Based on the fact that the Yang-Baxter relation reduces to the defining relation of the braid group, a general construction procedure of link polynomials has been established, starting from solvable statistical-mechanical models at criticality whose Boltzmann weights satisfy the Yang-Baxter relation [4].

We now know that both the "classic" Alexander polynomial [5] and the Jones polynomial are certain limits of a two-variable link polynomial called the *homfly* polynomial [6]. In this sense, the Alexander polynomial and the Jones polynomial are close relatives of each other. There is, however, a major difference between the two. The Alexander polynomial admits "multivariable" generalization such that each closed string constituting a link carries its own variable. Such a multivariable generalization is not known for the Jones polynomial. From the statistical-mechanical point of view, this type of multivariable link polynomial can be constructed from a multivariable solution of the Yang-Baxter relation. In fact, the recent state model construction of the multivariable Alexander polynomial made by Murakami [7] utilizes a "colored"-braid-group representation which is closely related to Felderhof's solution [8] of the Yang-Baxter relation for the free-fermion eight-vertex model [9].

The aim of the present Letter is clear: We construct a series of solutions of the colored Yang-Baxter relation in the braid limit. The solutions not only give a novel family of solvable lattice models but also serve as a basis for construction of a sequence of multivariable link invariants generalizing the multivariable Alexander polynomial.

Let us briefly review the case of the single-variable link polynomials [4], which is based on the closed-braid representation of links. We use the vertex-model terminology. By $\sigma_{kl,ij}(u)$ we denote the Boltzmann weight for the vertex configuration (kl,ij) as shown in Fig. 1. We have the following expression for the diagonal-to-diagonal transfer matrix $X_i(u)$ (u is the spectral parameter; $i=1,\ldots,n-1$):

$$X_i(u)=\sum_{rs,pq}\sigma_{rs,pq}(u)I^{(1)}\otimes I^{(2)}\otimes\cdots\otimes I^{(i-1)}\otimes E_{rp}\otimes E_{sq}\otimes I^{(i+2)}\otimes\cdots\otimes I^{(n)}, \quad (1)$$

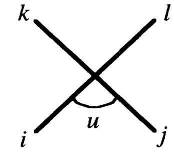

FIG. 1. Boltzmann weight $\sigma_{kl,ij}(u)$.

where E_{rp} is a matrix such that $(E_{rp})_{ij}=\delta_{ri}\delta_{pj}$, and the integer n (≥ 2) corresponds to the horizontal lattice size of the vertex model. The solvability condition for the vertex model, called the Yang-Baxter relation, reads

$$X_i(v)X_{i+1}(u+v)X_i(u)=X_{i+1}(u)X_i(u+v)X_{i+1}(v)\,. \quad (2)$$

Taking the limits $u,v\rightarrow\infty$ on both sides along a suitable direction in the complex u plane [4], Eq. (2) reduces to the following relation defining the braid group [$b_i=\lim X_i(u)$]:

$$b_ib_{i+1}b_i=b_{i+1}b_ib_{i+1}\,. \quad (3)$$

With the further relation $b_ib_j=b_jb_i$ (for $|i-j|\geq 2$), we have the n-string braid group B_n generated by $\{1,b_1,\ldots,b_{n-1}\}$. The explicit matrix form of b_i is given by Eq. (1) with $\{\sigma_{rs,pq}(u)\}$ replaced by its $u\rightarrow\infty$ limit $\{b_{rs,pq}\}$, which we call the *braid matrix*. By regarding b_i as the basic braiding operation between the ith and $(i+1)$th strings, we can identify braids as elements of the broad group.

For construction of link polynomials, two theorems are essential. One is Alexander's theorem [10], which guarantees that any link can be represented as a closed braid. The other is Markov's theorem [11], which states that two different braids representing a same link are transformed into each other by successive application of the *Markov moves:* $AB\rightarrow BA$ ($A,B\in B_n$) (type I), and $A\rightarrow Ab_n$ ($A\in B_{n+1}$, $b_n\in B_n$) (type II). Thus a link polynomial is a functional acting on the set of representations $\{B_n\}_{n=2}^{\infty}$, which has invariance with respect to the Markov moves. Namely, a functional $\alpha(\cdot)$ satisfying

$$\begin{aligned}&\alpha(AB)=\alpha(BA)\quad (A,B\in B_n)\,,\\ &\alpha(Ab_n)=\alpha(A)\quad (A\in B_n,\ b_n\in B_{n+1})\end{aligned} \quad (4)$$

is a link polynomial.

We regard Eq. (3) as a system of algebraic equations to determine the braid-matrix elements $\{b_{rs,pq}\}$. In ordinary noncolored cases, solutions contain only one nontrivial parameter which is often denoted by t. The resulting link invariant is a function of t. We now allow each string constituting a braid to have color, and associate different parameters to strings with different colors. Let us call the parameters *string variables*. Then Eq. (3) is generalized to the colored-braid relation (Fig. 2):

$$b_i(y,z)b_{i+1}(x,z)b_i(x,y)=b_{i+1}(x,y)b_i(x,z)b_{i+1}(y,z)\,, \quad (5)$$

where $b_i(x,y)$ represents the elementary braiding operation between a string with string variable x and that with y. The noncolored version Eq. (3) corresponds to $x=y=z=t$. In terms of the colored-braid matrix

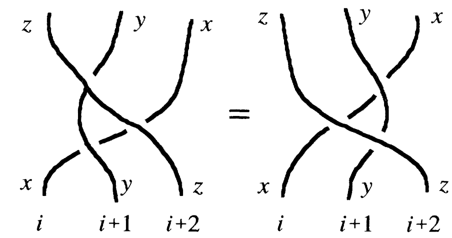

FIG. 2. Colored-braid relation.

$\{b_{kl,ij}(x,y)\}$, Eq. (5) reads

$$\sum_{\alpha\beta\gamma}b_{pq,\alpha\gamma}(y,z)b_{\gamma r,\beta k}(x,z)b_{\alpha\beta,ij}(x,y)=\sum_{\alpha\beta\gamma}b_{qr,\gamma\beta}(x,y)b_{p\gamma,i\alpha}(x,z)b_{\alpha\beta,jk}(y,z)\,. \quad (6)$$

We can regard Eq. (5) as a braid limit of the *colored* Yang-Baxter relation

$$X_i(y,z;v)X_{i+1}(x,z;u+v)X_i(x,y;u)=X_{i+1}(x,y;u)X_i(x,z;u+v)X_{i+1}(y,z;v)\,. \quad (7)$$

We should remark here that if we interpret the string variables x, y, and z as spectral parameters (or rapidities in the factorized S-matrix theory), we can regard the u-independent relation (5) as the standard Yang-Baxter relation. There is, however, a major difference between the known "ordinary" solutions and those given in this Letter. The ordinary solutions depend on the spectral parameters only in terms of their difference, which allows us to put $b_i(x,y)=X_i(x-y)$ and to write the Yang-Baxter relation in the familiar "additive" form (2). In our solutions, $X_i(x,y)$ is a *full two-variable function* of x and y, and we cannot write the Yang-Baxter relation (5) in the additive form. The nondifference property of the solution is interesting, giving a novel family of integrable vertex models. Moreover, the property is important from the knot-theoretical point of view; because of this property, we can have a consistent and nontrivial definition of colored braids where the string variables naturally represent colors of strings.

We focus on solutions of the u-independent relation (5). We assume the "charge-conservation condition" [12]: $b_{kl,ij}(x,y)=0$ if $i+j\neq k+l$. Let us introduce the following function ($m=0,1,2,\ldots$):

$$F_m(z,v)=\begin{cases}\prod_{k=0}^{m-1}(1-zv^k) & (m\geq 1),\\ 1 & (m=0).\end{cases} \quad (8)$$

For a general N-state model, the following form of $b_{kl,ij}(x,y)$ [$i,j,k,l\in\{-s,-s+1,\ldots,s\}$; $s=(N-1)/2$] solves (6):

$$b_{kl,ij}(x,y)=\begin{cases}[Q_{\bar k\bar j}(x\omega^{\bar i+\bar j},1/\omega)Q_{\bar i\bar l}(y\omega^{\bar i+\bar j},1/\omega)]^{1/2} & \text{for } i+j=k+l \text{ with } j+l\geq i+k\,,\\ 0 & \text{otherwise}\,,\end{cases} \quad (9)$$

where we have denoted $\bar{k}=s-k$ and

$$Q_{pm}(z,v)=\frac{F_p(v,v)F_p(zv,v)}{F_{p-m}(v,v)F_m(v,v)F_m(zv,v)}z^m v^{m^2},$$
$$\omega=\exp(2\pi i/N). \tag{10}$$

The inverse braid matrix is given by

$$[b^{-1}(x,y,\omega)]_{kl,ij}=[b(1/x,1/y,1/\omega)]_{lk,ji}. \tag{11}$$

We have checked the colored-braid relation up to $N=9$ by direct calculation. We believe that (9) satisfies (6) for general N, although the proof is not yet completed. We can find that for $3\le N\le 6$ the expression (9) with $x=y=t$ reproduces the noncolored solution found by Lee, Couture, and Schmeing [13]. For $N=2$, (9) gives a colored-braid matrix corresponding to the multivariable Alexander polynomial [7]. Using the braid matrices (9) and (11), we have an n-color representation of the n-string braid group. Note that after closing a braid to form a link, the number of independent colors N_c may be less than n. We denote the color of the ith string in the closed brain by $c(i)$ $(\in\{1,2,\dots,N_c\})$. The resulting closed-n-braid representation of a link contains N_c string variables $\{x_{c(1)},x_{c(2)},\dots,x_{c(n)}\}$. We denote the (n,N_c) representation of the colored-braid group by $B_n(\{x_{c(i)}\},\{c(i)\},N_c)$.

Let us discuss the link polynomials associated with the obtained braid-group representations. We first construct a functional called the Markov trace which we denote by $\phi(\cdot)$. As has been known for the noncolored case [4], construction of $\phi(\cdot)$ is reduced to finding a diagonal matrix $h=\mathrm{diag}(h_1,h_2,\dots,h_N)$ with the *Markov property*

$$\sum_{q=1}^{N} b_{pq,pq}(x,x)h_q=\xi \text{ (independent of } p),$$
$$\sum_{q=1}^{N} b^{-1}_{pq,pq}(x,x)h_q=\bar{\xi} \text{ (independent of } p), \tag{12}$$

where $\xi=\xi[b(x,x)]$, $\bar{\xi}=\xi[b^{-1}(x,x)]$ are constants. In the present case, h_p is given by

$$h_p=\omega^{-\bar{p}}, \tag{13}$$

with

$$\xi[b(x,x)]=1,\quad \xi[b^{-1}(x,x)]=1/x^{N-1}. \tag{14}$$

For convenience, we introduce the renormalized generators $\{g_i(x,y)\}$ by

$$g_i(x,y)=(1/x^{N-1}y^{N-1})^{1/4}b_i(x,y). \tag{15}$$

Corresponding braid matrices satisfy both the colored-braid relation and the Markov property with

$$\xi[g(x,x)]=\xi[g^{-1}(x,x)]=x^{-(N-1)/2}. \tag{16}$$

The renormalized generators define another (but an equivalent) colored-braid-group representation $\hat{B}_n(\{x_{c(i)}\},\{c(i)\},N_c)$. We introduce the H matrix by

$$H=h^{(1)}\otimes h^{(2)}\otimes\cdots\otimes h^{(k)}\otimes\cdots\otimes h^{(n)}, \tag{17}$$

where $h^{(i)}$ is the h matrix acting at the ith position. The unnormalized Markov trace $\phi(\cdot)$ is given by

$$\phi(A)=\mathrm{Tr}(HA)\quad [A\in\hat{B}_n(\{x_{c(i)}\},\{c(i)\},N_c)]. \tag{18}$$

Note that the functional $\phi(\cdot)$ is color independent, because the H matrix does not depend on the string variables $\{x_{c(i)}\}$. String variables appear only in the braid-group representation.

It is easy to see that the following quantity $\alpha(\cdot)$ has the Markov-move invariances (4) (with b_i's replaced by g_i's), and hence is a link polynomial:

$$\alpha(A)=\left[\prod_{i=1}^{n}x_{c(i)}^{(N-1)/2}\right]\phi(A)\quad [A\in\hat{B}_n(\{x_{c(i)}\},\{c(i)\},N_c)]. \tag{19}$$

We thus obtained a sequence $(N=2,3,4,\dots)$ of multivariable link polynomials.

For the noncolored case ($x_{c(i)}=t$, for all i) with $2\le N\le 6$, there has been an unproven conjecture [13] that the Markov trace $\phi(\cdot)$ with the h matrix (13) gives $\phi(A)=0$ [hence $\alpha(A)=0$] for any braid A. Since this property of ϕ originated from the tracelessness ($\sum_p h_p=0$) of the h matrix, we suppose that the colored version (18) and (19) should have the same property. The proposed regularization scheme to overcome the difficulty has been to modify the H matrix at its left edge: $h^{(1)}\to\mathrm{diam}(1,0,0,\dots)$ (Ref. [13]) or $h^{(1)}\to I$ (Ref. [14]). After the modification, however, the property $\phi(AB)=\phi(BA)$ [equivalently, $\alpha(AB)=\alpha(BA)$] becomes nontrivial. Validity of the modified functional for $N=2$ was proved recently [7], which gives support for the regularization scheme for general N.

In Ref. [14] we have pointed out a connection between the Alexander polynomial ($N=2$ case) and the Z_2-graded solution [15] of the Yang-Baxter relation. The connection can be simply seen from the classical limit $x,y\to 1$ of the braid matrix (9). In this limit, the element $b_{ji,ij}$ becomes

$$b_{ji,ij}(1,1)=\omega^{\bar{i}\bar{j}}. \tag{20}$$

If we assign a grade $p(i)$ to the index i as $p(i)=s-i$, (20) gives a Z_N analog of the graded permutation. Hence, we may call the solution (9) the Z_N-graded solution of the colored-braid relation (or colored Yang-Baxter relation in the braid limit). Physically, $b_{ji,ij}$ corresponds to the phase factor (or S matrix) with respect to the particle exchange $(i,j)\to(j,i)$. The form (20) indicates that the particles involved in our problem have fractional statistics. Recalling that the Z_2-graded solution corresponds to a free-fermion system, we can expect that our Z_N-graded solution describes statistical mechanics of anyons.

From the form of the solution (9), it is natural to examine the case where ω is replaced by other roots of unity: $\omega = \exp(2k\pi i/N)$ $(k=2,3,\ldots,N-1)$. A direct check of the colored-braid relation for $N \leq 8$ shows that not all values of k are admissible. The admissibility condition is, however, simple: *k and N are mutually prime.* This fact implies that (9) with $\omega = \exp(2k\pi i/N)$ gives a solution for (6) *for each rational number k/N.*

Our solution for the colored-braid relation presented in this Letter is the most symmetric one. We can discuss various transformation properties of the solution and their relation to the "fusion" solutions [16]. Although we have restricted ourselves to the braid relation (5), we can consider the u-dependent relation (7). Our braid matrices are also helpful in constructing the u-dependent solution. Details will be published elsewhere [17].

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