Mechanical Parametric Amplification and Thermomechanical Noise Squeezing

D. Rugar and P. Grütter[a]

IBM Research Division, Almaden Research Center, 650 Harry Road, San Jose, California 95120

(Received 20 May 1991)

A mechanical degenerate parametric amplifier has been devised which greatly increases the motional response of a microcantilever for small harmonic force excitations. The amplifier can improve force detection sensitivity for measurements dominated by sensor noise or backaction effects and can also produce mechanical squeezed states. In an initial squeezing demonstration, the thermal noise (Brownian motion) of the cantilever was reduced in one phase by 4.9 dB.

PACS numbers: 62.30.+d, 04.80.+z, 05.40.+j, 06.30.−k

Techniques that improve the sensitivity for detecting small forces are of interest in fields such as atomic force microscopy [1] and gravity wave detection [2]. Both attractive-mode force microscopes and resonant bar gravity wave detectors utilize a mechanical oscillator whose motion constitutes the signal of interest. Typically a measurement is made by first converting the mechanical motion into an electrical signal via a transducer and then amplifying the electrical signal. In some cases, backaction-evasion techniques can be employed to minimize (in one phase) the noise coupled back to the mechanical oscillator from the inherently noisy electrical amplifier [2–5].

In this paper, we explore a new force measurement strategy wherein the motion of the oscillator is mechanically preamplified by a large factor before transduction using a mechanical parametric amplifier. The amplifier incorporates a silicon microcantilever as a mechanically resonant element, operates in the degenerate mode, and is electrically pumped. A mechanical amplifier of this type is of fundamental interest because, in principle, degenerate parametric amplifiers can be noise-free down to the quantum-mechanical level [6,7]. In practical applications, mechanical preamplification can improve force detection sensitivity in cases where sensor noise or backaction effects dominate. In addition to its utility for amplifying subangstrom mechanical signals, the amplifier can be used to produce mechanical squeezed states. In particular, the amplifier has been used to produce a classical squeezed state where the thermal vibration (Brownian motion) of the microcantilever is reduced in one phase to an amplitude substantially smaller than the usual thermal equilibrium value. We believe that this is the first demonstration of thermomechanical noise squeezing.

The basic elements of the mechanical amplifier and associated measurement apparatus are shown in Fig. 1. A silicon microcantilever, similar to those developed for atomic force microscopy [8], is used as a mechanical resonator. The cantilever is 500 μm long, 10 μm wide, and several micrometers thick. It has a resonant frequency of 33.57 kHz, a spring constant on the order of 1 N/m, and a Q value greater than 10^4 in vacuum. The motion of the cantilever is monitored with 10^{-4}-Å/$\sqrt{\text{Hz}}$ sensitivity by an interferometer built from fiber-optic components [9]. For characterization purposes, such as gain measurement, the cantilever can be mechanically excited via a piezoelectric (PZT) bimorph.

To create a parametric amplifier effect, a means is required to periodically modulate (pump) some parameter of the mechanical oscillator, such as the spring constant. This is accomplished by positioning an electrode to within 50 μm of the cantilever and applying a time-varying voltage $V(t)$. The gradient of the electrostatic force from the electrode has the effect of modifying the spring constant according to

$$k(t)=k_0+k_p(t)\,, \tag{1}$$

where k_0 is the unperturbed spring constant and

$$k_p(t)=\frac{\partial F_e}{\partial x}=\frac{1}{2}\frac{\partial^2 C}{\partial x^2}[V(t)]^2\,. \tag{2}$$

F_e is the electrostatic force exerted on the cantilever by the electrode, x is the cantilever displacement, and C is the electrode-cantilever capacitance.

The behavior of mechanical parametric oscillators (such as the child's swing or the parametrically driven pendulum) have been studied since the nineteenth century [10,11]. The analysis of our system differs from this previous work in that the pump strength in our experiment is maintained below the threshold for self-sustained oscillation. The analysis we present below has much in common with the analysis of electrical parametric amplifiers [12].

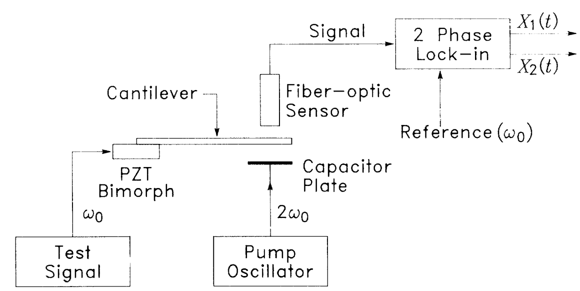

FIG. 1. Block diagram of the mechanical parametric amplifier and associated measurement apparatus. The spring constant of the cantilever is modified (pumped) at frequency $2\omega_0$ by the electric field from the capacitor plate.

We model the cantilever as a driven harmonic oscillator with time-varying spring constant. The equation of motion is

$$m\frac{d^2x}{dt^2}+\frac{m\omega_0}{Q}\frac{dx}{dt}+[k_0+k_p(t)]x=F(t)\,, \qquad (3)$$

where $x(t)$ is the cantilever displacement, $F(t)$ is an external driving force, Q is the quality factor of the resonance, and ω_0 is the resonant frequency of the cantilever. m, k_0, and ω_0 are related by $k_0=m\omega_0^2$.

Using the normal-mode approach described by Louisell [12] for electrical parametric amplifiers, we introduce the transformations

$$a=\frac{dx}{dt}+j\omega_1^*x\,, \qquad (4)$$

$$a^*=\frac{dx}{dt}-j\omega_1x\,, \qquad (5)$$

where $j=\sqrt{-1}$ and

$$\omega_1=\omega_0[(1-1/4Q^2)^{1/2}+j/2Q]\,. \qquad (6)$$

The inverse transformations are

$$x=\frac{a-a^*}{j(\omega_1^*+\omega)}\,, \qquad (7)$$

$$\frac{dx}{dt}=\frac{\omega_1a+\omega_1^*a^*}{\omega_1^*+\omega_1}\,. \qquad (8)$$

Substituting (7) and (8) into (3) and then performing a series of algebraic manipulations yields the simplified equation of motion

$$\frac{da}{dt}=j\omega_1a+j\frac{k_p(t)}{m}\frac{a-a^*}{\omega_1^*+\omega_1}+\frac{F(t)}{m}\,. \qquad (9)$$

We solve (9) in the limit of high Q for a driving force $F(t)$ at resonance given by

$$F(t)=F_0\cos(\omega_0t+\phi)\,. \qquad (10)$$

The pump voltage $V(t)$ is assumed to have the form $V(t)=V_0+V_p\sin2\omega_0t$. Substituting this into (2) and retaining only the $2\omega_0$ term, the time-varying spring constant is found to be

$$k_p(t)=\Delta k\sin2\omega_0t\,, \qquad (11)$$

where $\Delta k=(\partial^2C/\partial x^2)V_0V_p$.

We look for steady-state solutions of (9) which have the form

$$a(t)=A\exp(j\omega_0t)\,, \qquad (12)$$

where A is a complex constant. Substituting (10)-(12) into (9) and retaining the terms containing $\exp(j\omega_0t)$ (high-Q approximation), we find

$$\left[j(\omega_1-\omega_0)A-\frac{\Delta k}{2m(\omega_1^*+\omega_1)}A^*+\frac{F_0}{2m}e^{j\phi}\right]e^{j\omega_0t}=0\,. \qquad (13)$$

Noting that, to first order in $1/Q$, $\omega_1^*+\omega_1\approx2\omega_0$ and $\omega_1-\omega_0\approx j\omega_0/2Q$, we find

$$A=F_0\frac{Q\omega_0}{k_0}\left[\frac{\cos\phi}{1+Q\Delta k/2k_0}+j\frac{\sin\phi}{1-Q\Delta k/2k_0}\right]. \qquad (14)$$

If we write the cantilever motion as $x(t)=X_1\cos\omega_0t+X_2\sin\omega_0t$, then $X_1=\mathrm{Im}A/\omega_0$ and $X_2=\mathrm{Re}A/\omega_0$.

The gain of the amplifier is

$$G(\phi)=\frac{|X|_{\text{pump on}}}{|X|_{\text{pump off}}}=\frac{|A|_{\text{pump on}}}{|A|_{\text{pump off}}}\,, \qquad (15)$$

where $|X|=(X_1^2+X_2^2)^{1/2}$. Using (14) and the definition of Δk, the gain is given by

$$G(\phi)=\left[\frac{\cos^2\phi}{(1+V_p/V_t)^2}+\frac{\sin^2\phi}{(1-V_p/V_t)^2}\right]^{1/2}, \qquad (16)$$

where $V_t=2k_0/QV_0(\partial^2C/\partial x^2)$.

As expected, the gain of the amplifier is phase sensitive and is maximum for $\phi=\pi/2$. The gain for this phase increases with increasing pump strength and goes to infinity as $V_p\to V_t$. This is the threshold for parametric oscillation, a condition that is easily achieved in our experiments. To use the device as an amplifier, we operate with $V_p<V_t$. For the phase $\phi=0$, the gain is less than unity and decreases with increasing pump strength. The greatest deamplification that can be obtained in this phase without oscillation of the quadrature phase is achieved for $V_p\to V_t$, yielding $G(0)\to\frac{1}{2}$.

In the experiment, the amplifier gain is measured by driving the cantilever sinusoidally with the PZT bimorph and using the fiber-optic interferometer to observe the resulting cantilever oscillation. A two-phase lock-in amplifier is used to determine $|X|$ both with and without the pump voltage present. Typical values of $|X|_{\text{pump off}}$ were on the order of 1 nm. The cantilever Q value was approximately 1×10^4, obtained in a vacuum of 9×10^{-2} torr. The pump parameters were $V_0=10$ V and V_p in the range of 0-2.5 V.

Figure 2(a) shows a comparison of the measured and theoretical amplifier gain as a function of pump voltage V_p. The theory in (16) is fitted to the experimental data using $V_t=2.6$ V. Excellent agreement between theory and experiment is obtained for both the amplified and deamplified phase. The plot shows experimental gain values up to 25. Gain values as high as 100 could be readily achieved, although precise measurement was difficult since the amplifier is very close to spontaneous oscillation at such high gain.

The experimental and theoretical phase response is shown in Fig. 2(b). The experiment was performed using $V_p=2.5$ V. The theory was fitted using this value of V_p and a threshold voltage $V_t=2.65$ V. Again, the agreement between theory and experiment is excellent.

A mechanical amplifier of the type described above may provide a significant advantage for detecting small harmonic forces when the measurement is dominated by

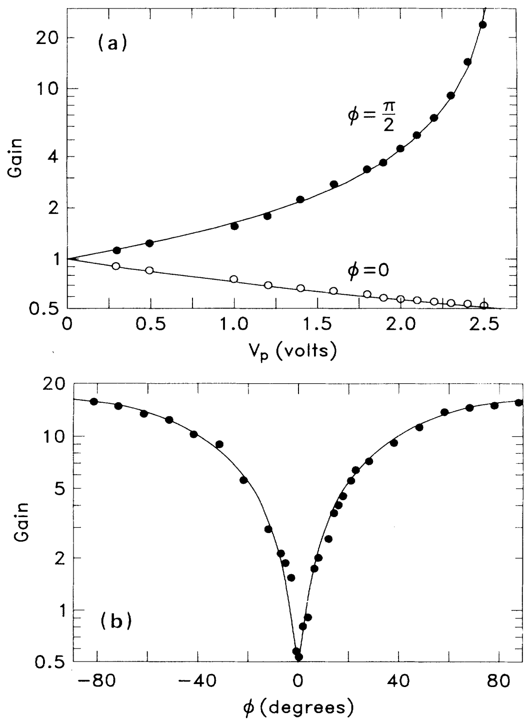

FIG. 2. (a) Parametric amplifier gain as a function of pump voltage. Results for both the amplified and deamplified signal phases are shown. (b) Phase dependence of the gain. In both (a) and (b), the points are experimental values and the curves are from the theory in Eq. (16).

sensor noise or backaction-induced noise. Consider the case of a cantilever harmonically driven at resonance by a small signal force and monitored in a fixed narrow bandwidth by an interferometer. A signal force of the form in (10) gives rise to cantilever motion $|X| = GQF_0/k_0$. If shot noise of the interferometer is the dominant noise source, then the uncertainty in cantilever position is [13]

$$\langle (\Delta x)^2_{\rm shot} \rangle = (hc\lambda/8\pi^2 I_0)\Delta f\,, \qquad (17)$$

where I_0 is the optical power, λ is the wavelength, and Δf is the measurement bandwidth. For unity signal-to-noise ratio [i.e., $|X| = \langle (\Delta x)^2_{\rm shot}\rangle^{1/2}$], the minimum detectable signal force is found to be inversely proportional to the amplifier gain and given by

$$F_{0,\min} = \frac{k_0}{GQ}\left(\frac{hc\lambda}{8\pi^2 I_0}\Delta f\right)^{1/2}. \qquad (18)$$

In addition to shot noise, the interferometer measurement also generates a backaction-noise force due to the statistics of photons impinging on the cantilever. This backaction force is given by [13]

$$\langle (\Delta F)^2_{\rm ba}\rangle = 4I_0 h\Delta f/c\lambda\,. \qquad (19)$$

For a narrow bandwidth about resonance, this random force results in the cantilever motion

$$\langle (\Delta x)^2_{\rm ba}\rangle = (G^2Q^2/k_0^2)\langle (\Delta F)^2_{\rm ba}\rangle\,. \qquad (20)$$

The force measurement is optimized by choosing the optical power I_0 so as to minimize $\langle (\Delta x)^2_{\rm shot}\rangle + \langle (\Delta x)^2_{\rm ba}\rangle$. The optimum value of I_0 is found to be $I_0 = k_0 c\lambda/4\sqrt{2}\pi GQ$ and the minimum detectable force is

$$F_{0,\min} = (\sqrt{2}hk_0\Delta f/\pi QG)^{1/2}. \qquad (21)$$

Whether considering shot noise alone or both shot-noise and backaction effects, the results in (18) and (21) clearly demonstrate that force sensitivity can be significantly improved by using a mechanical amplifier with high gain. The advantage of mechanical preamplification is a general one and is valid for other measurement techniques besides interferometry. With respect to backaction effects, the essential point is that mechanical preamplification increases the mechanical signal so that the coupling efficiency of the transducer can be reduced without net loss of transducer signal-to-noise ratio; the backaction-noise force is smaller as a result of the reduced transducer coupling efficiency. In this sense, mechanical preamplification is an alternative to backaction evasion.

The preceding discussion neglected the thermal vibration of the cantilever, an assumption that would be valid at sufficiently low temperatures. When thermal vibration is the dominant noise source, as was the case in our room-temperature experiments, then mechanical preamplification yields no net improvement in force sensitivity. This is because the signal and thermal vibrations are amplified by equal amounts. Nevertheless, the room-temperature device is interesting because of its ability to produce mechanical squeezed states.

In a narrow bandwidth, the thermal noise can be divided into quadrature phases $x(t) = X_1(t)\cos\omega_0 t + X_2(t) \times \sin\omega_0 t$, where $X_1(t)$ and $X_2(t)$ are random variables that vary slowly compared to ω_0. $X_1(t)$ and $X_2(t)$ were measured by the two-phase lock-in amplifier (time constant $=0.3$ sec) and the results sampled every 0.1 sec by a digital oscilloscope.

We plot in Fig. 3(a) sampled values of X_1 vs X_2 taken with the pump off (i.e., at thermal equilibrium). As expected, the measured noise distribution is independent of phase [i.e., the distribution of X_1 vs X_2 is circularly symmetric in Fig. 3(a)]. A histogram of X_2 values is shown in Fig. 3(b), along with the Gaussian curve that is the best fit to the histogram. The variance of X_2 was found to be 0.053 Å^2. The noise measurement was then repeated with the pump turned on. Figure 3(c) shows the result with the amplifier operating such that the amplified phase experiences a gain of about 4.7 and the deamplified phase has gain of 0.56. As a result of the phase sensitivity of the amplifier, the noise has been reduced for X_2 at the expense of increased noise in X_1. In other words, the thermal vibration noise has been squeezed. The paramet-

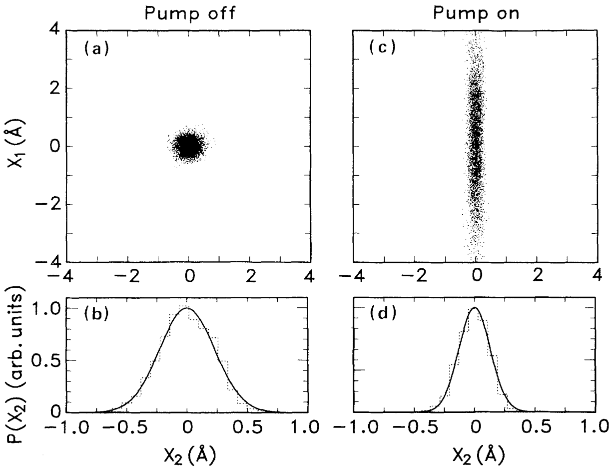

FIG. 3. (a) Thermal vibration noise measured with the pump turned off (thermal equilibrium) and plotted in the X_1-X_2 phase plane. The circularly symmetric distribution is the result of 4000 measurements taken every 0.1 sec. (b) The histogram shows the relative probability of occurrence for the X_2 values shown in (a). The smooth curve is the best-fit Gaussian with 0.053-$Å^2$ variance. (c) Thermal noise measured with the amplifier pump turned on. As a result of the amplifier phase dependence, the noise distribution has been squeezed. (d) Histogram and best-fit Gaussian for the X_2 values in (c). The squeezing effect reduces the X_2 variance to 0.017 $Å^2$.

ric amplifier has reduced the X_2 variance to 0.017 $Å^2$, a 4.9-dB reduction relative to the thermal equilibrium value. This variance corresponds to the thermal equilibrium noise one would measure if the cantilever were cooled to 96 K.

To the authors' knowledge, this result is the first demonstration of thermomechanical noise squeezing. This effect is the classical mechanical analog of quantum noise squeezing, which has been demonstrated using parametric amplifiers [14,15] and other techniques [16,17] for electromagnetic waves at both optical and microwave frequencies. Mechanical noise squeezing may eventually be useful for preparing mechanical resonators (such as gravity wave detectors) in a low-noise initial state. Since the deamplification achievable without parametric oscillation is limited to $G(0)=\frac{1}{2}$, the noise squeezing from our device is presently limited to -6 dB. A similar squeezing limit was found for intracavity fields in optical parametric amplifiers [18,19].

One of the authors (D.R.) is especially grateful to M. D. Levenson for introducing him to the subject of squeezed states. The authors also thank H. Birk, P. Guethner, H. J. Mamin, and P. Wimmer for their contributions to the experimental apparatus and M. Bocko, U. Dürig, and R. Shelby for enlightening discussions. The microcantilever was provided by O. Wolter, Th. Bayer, and J. Greschner, IBM-GMTC.

[a]Present address: Institut für Physik, Universität Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland.

[1] G. Binnig, C. F. Quate, and Ch. Gerber, Phys. Rev. Lett. **56**, 930 (1986).
[2] C. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).
[3] V. B. Braginsky, Y. I. Vorontsov, and K. S. Thorne, Science **209**, 547 (1980).
[4] M. F. Bocko and W. W. Johnson, Phys. Rev. Lett. **48**, 1371 (1982).
[5] F. Bordoni and R. Onofrio, Phys. Rev. A **41**, 21 (1990).
[6] H. Takahasi, in *Advances in Communication Systems,* edited by A. V. Balakrishnan (Academic, New York, 1965), Vol. 1.
[7] C. M. Caves, Phys. Rev. D **26**, 1817 (1982).
[8] O. Wolter, Th. Bayer, and J. Greschner, J. Vac. Sci. Technol. B **9**, 1353 (1991).
[9] D. Rugar, H. J. Mamin, and P. Guethner, Appl. Phys. Lett. **55**, 2588 (1989).
[10] Lord Rayleigh (J. W. Strutt), Philos. Mag. **24**, 145 (1887).
[11] W. W. Mumford, Proc. IRE **48**, 848 (1960).
[12] W. H. Louisell, *Coupled Mode and Parametric Electronics* (Wiley, New York, 1960), Chap. 4.
[13] W. A. Edelstein, J. Hough, J. R. Pugh, and W. Martin, J. Phys. E **11**, 710 (1978).
[14] L. A. Wu, M. Xiao, and H. J. Kimble, J. Opt. Soc. Am. B **4**, 1465 (1987).
[15] R. Movshovich, B. Yurke, and P. G. Kaminsky, Phys. Rev. Lett. **65**, 1419 (1990).
[16] R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985).
[17] R. M. Shelby, M. D. Levenson, S. H. Perlmutter, and R. G. DeVoe, Phys. Rev. Lett. **57**, 691 (1986).
[18] G. Milburn and D. F. Walls, Opt. Commun. **39**, 401 (1981).
[19] M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).