Spectra of Velocity and Temperature Fluctuations with Constant Entropy Flux of Fully Developed Free-Convective Turbulence

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It is shown that the frequency-temperature spectrum recently observed by Wu *et al.* in developed convective turbulence, $F_{TT}(\omega) \propto \omega^{-1/4}$, follows from the condition that the *entropy flux in k space is constant,* just as the Kolmogorov-Obukhov spectrum of barotropic turbulence, $F_{VV}(k) \simeq (\varepsilon/\rho)^{2/3} k^{-11/3}$, follows from the condition that *kinetic-energy flux* $\varepsilon(k) = \varepsilon$. On the contrary, for convective turbulence $\varepsilon(k)$ changes as $k^{-4/5}$ because of conversion of kinetic energy into a potential energy, which leads to a stronger k dependence of the double velocity moment $[F_{VV}(k) \propto k^{-21/5}]$ than that for barotropic turbulence.

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Recently, the turbulent motion of helium gas in convective cells heated from below at very high Rayleigh numbers Ra (up to 4×10^{14}) was studied [1]. The flow in the center zone of the cell (without mixing zone and boundary layer) is homogeneous and, as is believed, characteristic of free-convection flow. The frequency power spectrum of thermal fluctuations at the center of the cell $F_{TT}(\omega)$ reflects properties of this flow. As Ra increased above 10^8, the power spectrum acquired the form

$$F_{TT}(\omega) \propto \omega^{-\alpha}, \quad \alpha = 1.4 \pm 0.05 , \tag{1}$$

in some region of ω: $\omega_{\min} < \omega < \omega_{\max}$. At $\mathrm{Ra} \simeq 7\times10^{10}$, $\omega_{\max}/\omega_{\min} \simeq 30$ [1].

The physical concept of fully developed free-convective turbulence given in this Letter explains spectrum (1). Let me briefly recall the Kolmogorov-Obukhov (KO) concept of developed barotropic turbulence [2], which is based on Richardson's cascade model with the largest eddies produced by the forces driving the flow. Being unstable, these eddies of size L_{ext} split into smaller ones, producing unstable eddies again and again. The last-generation eddies of size $L_{\text{int}} \ll L_{\text{ext}}$ are still stable and can only dissipate due to viscosity. Assuming that in this cascade all statistical details related to the source of energy, except for the energy flux ε in k space, are lost, KO found the following expression for the double velocity moment in the inertial interval of scales ($L_{\text{ext}} > 1/k > L_{\text{int}}$):

$$F_{VV}(k) = c_{\text{KO}} (\varepsilon/\rho)^{2/3} k^{-11/3} . \tag{2}$$

In this Letter we will show that spectrum (1) can also be derived from Richardson's cascade model [2]. However, it is another integral of motion, namely entropy, whose flux determines the inertial interval part of the spectra.

Let us consider a free-convective turbulence at Rayleigh number $\mathrm{Ra} = \beta g \Delta L^3/\nu\kappa \gg 1$, with β being the volume expansion coefficient, g the gravitational acceleration, L the size of the system (central zone of a box), Δ the temperature fluctuation, and, finally, ν and κ the kinematic viscosity and the thermal diffusivity, respectively.[3] Because of buoyant acceleration, temperature fluctuations produce L eddies with some velocity $\mathbf{v}$. At $\mathrm{Ra}_c \gg 1$, their Reynolds number $\mathrm{Re}(L) = vL/\nu \gg 1$, which means that they are unstable. The L eddies cause Richardson's cascade of decays which gives rise to eddies of different sizes l in the inertial interval. In the field of the L-eddies' temperature gradient the turbulent velocity field $\mathbf{v}(\mathbf{r},t)$ produces temperature fluctuations of various sizes l which interact with eddies of the same size because of the buoyancy effect. It is important that eddies of different sizes can exchange different amounts of energy with the gravitational field. As a result, the energy flux $\varepsilon(k)$ being transferred from k eddies to those of smaller size will no longer be constant [see (18)]. Therefore it is quite expected that the spectrum of convective turbulence does differ from the KO one.

Conservation laws.— We consider fluid behavior governed by the usual equations for free convection [4]:

$$\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] + \nabla p/\rho - \mathbf{g} = \nu \Delta \mathbf{v} , \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0 , \tag{4}$$

$$\frac{\partial(\rho s)}{\partial t} + \operatorname{div}\mathbf{j} = \kappa |\nabla T|^2/T^2 + \frac{\eta}{2T}\left[\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_e}{\partial x_e}\right]^2 , \tag{5}$$

$\mathbf{j} = \rho s \mathbf{v} - \kappa \nabla T/T$ and $\eta = \nu\rho$. Here s is the entropy of unit mass and $\mathbf{j}$ is the entropy flux in the $\mathbf{r}$ space. In barotropic turbulence the fluid temperature $T(\mathbf{r},t)$ and the density of entropy $\rho s(\mathbf{r},t)$ are constant and Eq. (5) becomes trivial. In the inertial interval (where energy pumping and dissipation are not essential), within the Euler equation the total energy

$$E = (\rho/2) \int |\mathbf{v}(\mathbf{r},t)|^2 d\mathbf{r} = \int E(k,t)\, dk \tag{6}$$

is conserved in the incompressible limit. Therefore $E(k,t)$ may be written in the divergent form

$$\partial E(k,t)/\partial t + \partial \varepsilon(k,t)/\partial k = 0 . \tag{7}$$

Thus the stationary condition $\partial E(k,t)/\partial t = 0$ is equivalent to the well-known statement that *for barotropic turbulence the kinetic-energy flux $\varepsilon(k,t)$ does not depend on k.*

In the presence of temperature fluctuations Eq. (5) becomes nontrivial. It describes the conservation law of the total entropy S of the system in the inertial interval [4]:

$$dS(t)/dt=0, \quad S(t)=\int \rho s\, d\mathbf{r}. \tag{8}$$

We shall assume that the temperature fluctuations are small:

$$T'(\mathbf{r},t)\equiv T(\mathbf{r},t)-T_0 \ll T_0 \equiv \int \langle T(\mathbf{r},t)\rangle d\mathbf{r} \Big/ \int d\mathbf{r}.$$

(T_0 is the mean temperature of the system and the angular brackets denote ensemble averaging.) Then the density of entropy (ρs) can be expanded into a series in T':

$$\rho s=\rho_0 s_0+[\partial(\rho s)/\partial T]_p T' - bT'^2/2+\cdots. \tag{9}$$

Here $b=-[\partial^2(\rho s)/\partial T^2]_{pp}>0$. The first term is not involved in the equation of motion and may be omitted. The second term in not essential in a statistical sense since in the inertial interval $\langle T'\rangle=0$. Thus, the first nontrivial term in (9) is proportional to T'^2. Terms omitted in (9) (which are $\propto T'^3, T'^4, \ldots$) are small. As a result, one has the following equation for the total mean entropy:

$$\langle S(t)\rangle \simeq \tfrac{1}{2} b \int \langle T'(\mathbf{r},t)^2\rangle d\mathbf{r} = \tfrac{1}{2} b \int F_{TT}(\mathbf{k},t)\, d\mathbf{k}. \tag{10}$$

Here $F_{TT}(\mathbf{k},t)$ is the simultaneous pair correlator of temperature fluctuations in the k representation. Equation (10) allows one to introduce *an entropy density in k space,*

$$S(k,t)=(bk^2/2)\int F_{TT}(k,\Omega,t)\, d\Omega, \tag{11}$$

in such a way that the total entropy $S(t)$ is $\int S(k,t)dk$ [in (11) Ω is solid angle]. In the theory of turbulent convection the entropy density $S(k,t)$ plays the same role as the energy density $E(k,t)$ in the theory of barotropic turbulence. By analogy with (7) one can obtain *a continuity equation for entropy density* in the inertial interval:

$$\partial S(k,t)/\partial t+\partial\mu(k,t)/\partial k=0. \tag{12}$$

Here $\mu(k,t)$ is *entropy-density flux* in k space. In the stationary case $S(k,t)$ is independent of time. According to (12) the flux μ has to be independent of k in the inertial interval (like the energy-density flux ε for barotropic turbulence). Note that according to Eqs. (9)–(12) $\mu(k) \simeq 2\pi bN(k)$, where N is the *influx of the mean "fluctuation intensity"* $\langle T'^2/2\rangle$ which was introduced in the theory of convective turbulence by Boldgiano [5] and Obukhov [6]. They also assumed that $T' \ll T_0$ and started from Eqs. (3)–(5), written in the Boussinesq approximation (BA). However, in the BA, Eq. (5) conserves not only $\int T'^2(\mathbf{r},t)d\mathbf{r}$ but also $\int \Phi(T'(\mathbf{r},t))d\mathbf{r}$, where Φ is an arbitrary function of T'. Therefore it is not quite clear what function Φ determines the turbulence spectra. Only a single function—the total entropy of the system $S(t)$—has physical meaning beyond the framework of the BA.

Dimensional analysis of the problem.—Let us now list the dimensional parameters which affect the velocity and the temperature fluctuations of free-convective turbulence in the inertial interval: $L_{\text{int}} \ll l \ll L_{\text{ext}}$. We believe that viscosity ν and thermal diffusivity κ are not essential due to the inequality $l \gg L_{\text{int}}$. We also assume that in the cascade the process of entropy transfer between eddies with different scales is local and all statistical details related to a drain of entropy on the largest scale L_{ext} are lost except for the entropy-density flux μ. Besides μ (or N) the list of essential parameters includes the wave vector k and (in the BA) the factor βg. Using μ, βg, and k the double moments of the v and T' fields can be constructed only as follows:

$$\begin{aligned} F_{VV}(k)&=c_{VV}(G^2\Gamma)^{2/5}k^{-21/5},\\ F_{TT}(k)&=c_{TT}T_0^2(\Gamma^2/G)^{2/5}k^{-17/5},\\ F_{TV}(k)&=c_{TV}T_0(\Gamma^3 G)^{1/5}k^{-19/5}.\end{aligned} \tag{13}$$

Here c_{VV}, c_{TT}, and c_{VT} are the dimensionless universal factors which do not depend on the condition of turbulence excitation, $G\equiv\beta g T_0$ is a characteristic acceleration, and $\Gamma\equiv 2\pi\mu/\rho c_p$ is the entropy flux in the system.

Physical picture of the free-convective turbulence.—Equations (13) give

$$\begin{aligned} v^2(l)&=\langle|\mathbf{v}(\mathbf{r}+\mathbf{l},t)-\mathbf{v}(\mathbf{r},t)|^2\rangle \simeq (G^2\Gamma)^{2/5}l^{\mu_{VV}},\\ T'^2(l)&=\langle[T'(\mathbf{r}+\mathbf{l},t)-T'(\mathbf{r},t)]^2\rangle \simeq T_0^2(\Gamma^2/G)^{2/5}l^{\mu_{TT}},\\ \langle[v_z(\mathbf{r}+\mathbf{l},t)&-v_z(\mathbf{r},t)][T'(\mathbf{r}+\beta\mathbf{l},t)-T'(\mathbf{r},t)]\rangle \\ &\simeq T_0(\Gamma^3 G)^{1/5}l^{\mu_{TV}},\\ \mu_{VV}&=\tfrac{6}{5},\quad \mu_{TT}=\tfrac{2}{5},\quad \mu_{TV}=\tfrac{4}{5}.\end{aligned} \tag{14}$$

The scaling indices μ_{VV} and μ_{TT} were obtained by Procaccia and Zeitak [7] in the dynamical approach. Relations (14) allow one to estimate (I) the characteristic velocity $v(l)$ of l eddies (eddies of size l), (II) their rotation frequency $\gamma(l)\simeq v(l)/l$ and lifetime $\tau(l)\simeq 1/\gamma(l)$, (III) their *amplitude of temperature fluctuation* $T'(l)$, and, finally, (IV) *current Rayleigh* and *current Reynolds numbers for l eddies:*

$$\begin{aligned}\mathrm{Ra}(l)&=\beta g T'(l)l^3/\nu\chi \simeq (\Gamma G^2)^{2/5}l^{16/5}/\nu\chi,\\ \mathrm{Re}(l)&=v(l)l/\nu \simeq (\Gamma G^2)^{1/5}l^{8/5}/\nu.\end{aligned} \tag{15}$$

In our rough estimations we took the Prandtl number (defined as $P=\nu/\chi$) equal to unity. Then $\mathrm{Re}^2(l) \simeq \mathrm{Ra}(l)$. Now from the relation $\mathrm{Ra}(L_{\text{int}})\simeq \mathrm{Ra}_{\text{cr}}\simeq 1$ [which is the same as $\mathrm{Re}(L_{\text{int}})\simeq \mathrm{Re}_{\text{cr}}\simeq 1$], one can estimate the internal (*dissipation*) scale:

$$L_{\text{int}} \simeq L_{\text{ext}}(\mathrm{Ra}_{\text{cr}}/\mathrm{Ra})^{5/16} \simeq L_{\text{ext}}(\mathrm{Re}_{\text{cr}}/\mathrm{Re})^{5/8}. \tag{16}$$

It is interesting to compare estimates (15) and (16) for convective turbulence with the similar ones for barotropic turbulence:

$v(l) \propto l^{1/3}$, $\gamma(l) \propto l^{-2/3}$, and $L_{\text{int}} \simeq L_{\text{ext}}(\text{Re}_{\text{cr}}/\text{Re})^{3/4}$.

Let us now find the χ dependence of *the rate of entropy production* $\Gamma_{\text{pr,th}}(\chi)$ due to thermal diffusivity. Using (5), (9), and (13) we obtain

$$\Gamma_{\text{pr,th}} = 2\pi\chi \int [|\nabla T|^2/T^2] d\mathbf{r}$$
$$\simeq 2c_{TT}\chi(\Gamma^2/G)^{2/5} \int k^{3/5} dk$$
$$\simeq \chi(\Gamma^2/G)^{2/5} k_{\max}^{8/5} .$$

The integral diverges at $k \to \infty$ and has to be cut off at $k_{\max} \simeq 1/L_{\text{int}}$. Substituting L_{int} from (16) we find that $\Gamma_{\text{pr,th}}$ does not depend on χ and equals the entropy flux in the system $\Gamma \equiv 2\pi\mu/\rho c_p$. *It is a stationary condition (for entropy) which shows that our considerations are self-consistent.* At the same time Γ has to be equal to *the rate of entropy extraction* from the system Γ_{ext}. The decrease in entropy is due to nonequilibrium processes which excite convective turbulence (like mixing of cold and hot fluids, heating of a fluid from below, and so on [4]). Following KO we assumed that all details of these processes are unimportant in the inertial interval except for $\Gamma_{\text{ext}} = \Gamma$ which determines the level of turbulence excitation via (13). It is also interesting to estimate the rate of entropy production due to viscosity:

$$\Gamma_{\text{pr,v}} \simeq (\nu/c_p T_0) \int k^2 F_{VV}(k) dk$$
$$\simeq (\nu/c_p T_0)(G^2\Gamma)^{2/5} k_{\max}^{4/5}$$
$$\simeq \Gamma (GL^2/\nu)^2 / c_p T_0 \text{Ra}^{5/4} .$$

The ratio $\Gamma_{\text{pr,v}}/\Gamma_{\text{pr,th}}$ tends to zero at $\text{Ra} \to \infty$. So, *the main process of entropy production is thermal diffusivity.*

Let us now find the k dependence of the energy flux $\varepsilon(k)$. From (6) it follows that

$$\partial\langle v(0,t)v(r,t)\rangle/\partial t \simeq \langle v(v\cdot\nabla)v\rangle \simeq v^3(r)/r .$$

Thus

$$\frac{\partial}{\partial t} F_{VV}(\mathbf{k},t) \simeq \int \frac{v^3(r)\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}}{r} \simeq \frac{v^3(1/k)}{k^2} . \quad (17)$$

By comparing (17) and (6) and (7) we have $\varepsilon(k) \simeq \rho k v^3(1/k)$ and using the estimate $v(l) \propto l^{-1/3}$ for barotropic turbulence, we obtain that $\varepsilon(k)$ is constant. However, for convective turbulence one can obtain, using (15),

$$\varepsilon(k) \simeq \rho(G^2\Gamma)^{3/5} k^{-4/5} . \quad (18)$$

The buoyancy term $-\beta g T'$ leads to an additional term in the equation of continuity for kinetic energy (7):

$$\frac{\partial E(k,t)}{\partial t} + \frac{\partial \varepsilon(k,t)}{\partial k} = -\rho\beta g k^2 \int F_{TV}(\Omega,k) d\Omega \equiv I_{TV}(k) . \quad (19)$$

Using (13) we have $I_{TV}(k) \simeq \rho(G^2\Gamma)^{3/5} k^{-9/5}$, which equals (within our accuracy) $\partial\varepsilon(k)/\partial k$. Thus, Eq. (19) for $\partial E/\partial t = 0$ is the *stationary condition for the kinetic energy of k eddies.*

The next question is the following: What is the direction of energy fluxes? It is known that in thermodynamic equilibrium [when $F_{VV}(\mathbf{k}) = T_0/\rho$ and does not depend on k] the energy flux $\varepsilon(k)$ is zero. If $F_{VV}(\mathbf{k})$ decreases with increasing k, the interaction between eddies leads to the equilibrium value $F_{VV}(\mathbf{k}) = \text{const}$. Therefore, the energy flows from small k (where F_{VV} is large), which is valid not only for the KO spectrum (2) but also for our spectra (13). As a result, for free-convective turbulence $\varepsilon(k) > 0$ and $\partial\varepsilon/\partial k < 0$. Then from (19) it follows that $I_{TV}(k) < 0$. This means that the kinetic energy of k eddies *is lost* through work done *against* buoyancy forces in the inertial interval of scales, i.e., *kinetic energy is converted into potential energy of the fluid,* controlled by the temperature. For this reason the kinetic-energy power spectrum $F_{VV}(\mathbf{k})$ of the convective turbulence (13) decreases faster with increasing k than the KO spectrum (2) with $\varepsilon(k) = \text{const}$. Therefore, spectra (13) initially were suggested in [5] and [6] *only* for mechanically excited turbulence in a fluid with stable thermal stratification. (Note, that such a statement is valid for only a very narrow range of turbulence-excitation parameters [8].) But for convective turbulence in a fluid heated from below all the authors of previous studies [2,5,6] believe that l eddies (at any l) will *draw additional kinetic energy from potential energy.* If so, the spectrum $F_{VV}(k)$ has to decrease *more slowly* with increasing k than the KO spectrum (2). However, the dimensional analysis yields *only* the spectra (13). An explanation of this paradox follows from *the locality of dynamic interaction of eddies.* Locality means that the main effect on the behavior of l eddies (except of their sweeping in the turbulent velocity field) is exerted by their interaction with the l' eddies of the same order of size. (A proof of this statement will be given in [9].) Stable and unstable stratifications differ only in long-scale temperature gradients, which is not essential because of interaction locality. Therefore, the same physical processes, which conserve the entropy in the inertial interval, are responsible for the turbulent behavior both for stable or unstable stratifications and for free-convective turbulence (without temperature stratification). Thus, in particular, *the spectra (13) have to describe the convective turbulence in a box heated from below* [1].

To compare the observed frequency spectrum $F_{TT}(\omega)$ (1) with the predicted momentum spectrum $F_{TT}(k)$ (13), one can use *a sweeping relation in the reference system where* $\langle v(\mathbf{r},t)\rangle = 0$ [10]:

$$F(\omega) \simeq k^2 F(k)/V_t \text{ at } k = \omega/V_t\,. \tag{20}$$

This looks like Taylor's hypothesis of *frozen turbulence* but contains $V_t = \langle |v(\mathbf{r},t)|^2\rangle^{1/2}$ instead of the mean velocity of flow, which equals zero. Sweeping relation (20) means that the main contribution to the variations of temperature and velocity signal (on a frequency ω in a fixed point of a box) is made by sweeping of l eddies with $l \simeq V_t/\omega$ in the turbulent velocity field V_t. We neglect in (20) the shape variation of l eddies during their *sweeping time* $\tau_s = l/V_t$, which is possible for $\tau(l) \gg \tau_s$. So, an *applicability parameter of the sweeping relation* (20) is $\xi(k) = 1/\tau(l)kV_t \ll 1$, $l \simeq 1/k$.

Using (15) we have $\xi(k) \simeq (kL_{\rm ext})^{-3/5}$ or $\xi(\omega) \simeq (\omega/\omega_{\max})^{3/5}$ for convective turbulence which is interesting to compare with $\xi(k) \simeq (kL_{\rm ext})^{-1/3}$ for barotropic turbulence. At $F(k) \propto k^{-\eta}$ Eq. (20) gives $F(\omega) \propto \omega^{-\alpha}$, $\alpha = \eta - 2$. Substituting $\eta_{TT} = \frac{17}{5}$, we have $F_{TT}(\omega) \propto \omega^{-7/5}$. In more detail, using (13) and (20) one can obtain

$$F_{TT}(\omega) \simeq \frac{\Delta^2}{\omega_{\max}} \left[\frac{\omega_{\max}}{\omega}\right]^{7/5}, \quad \omega_{\max} \simeq \frac{V_T}{L_{\rm ext}}\,. \tag{21}$$

The scaling index $\alpha = \frac{7}{5}$ is in excellent agreement with the experimental spectra (1). We think it means that *the cascade spectrum of free-convective turbulence with constant entropy flux was observed in [1].* Additional support for this statement comes from a very simple and clear picture of the phenomenon given above. Note, that at $\mathrm{Ra} > 7\times 10^{10}$ another power spectrum (with $\alpha \simeq 2.4$) arises at high frequencies [1]. It shows that turbulence at high frequencies cannot be considered as *a free turbulence* because of the mixing-zone effect. Another viewpoint on the experimental spectra (1) was presented by Castaing [11]. He uses the KO spectrum (2) for the *velocity field* in *barotropic* turbulence, but does not consider that $\varepsilon(k)$ depends on k for *free-convective* turbulence. Also, in [11] the theoretical plot for $E(k/k_0) \propto k^2 F_{VV}(k/k_0)$ has been compared with the experimental plot for $F_{TT}(\omega)$ without any explanation of the relation between these two different functions. In light of the considerations presented above the approach of [11] seems to me rather doubtful.

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