Exact Solution of the Perk-Schultz Model

H. J. de Vega and E. Lopes
Laboratoire de Physique Theorique et Hautes Energies, Tour 16, Universite Paris VI, 4, Place Jussieu, 75252 Paris CEDEX 05, France
(Received 22 April 1991)

In the Perk-Schultz vertex model links can have $n+1$ different colors and the weights are ferroelectric or antiferroelectric depending on a discrete parameter $\epsilon_a = \pm 1$ $(1 \le a \le n)$. We compute the exact ground-state spins and find ferrielectric behavior for general signs ϵ_a. The exact free energy and excitation spectrum are found as well as the finite-size corrections yielding the central charge and conformal dimensions. The scaling limit yields a quantum field theory whose mass spectrum and S matrix are explicitly obtained through the light-cone approach. The mass spectrum surprisingly depends on the anisotropy parameter and the signs of ϵ_a.

PACS numbers: 75.10.Hk, 11.10.−z, 75.50.Gg, 77.80.−e

Current interest in integrable two-dimensional lattice models comes from their exact solvability and from the remarkable property that they provide representatives for all known critical behaviors [1,2], that is, solvable representations for all universality classes.

As is known, the solutions of the Yang-Baxter equations (that provide all integrable models) are classified by Lie algebras ($\mathcal{G}$) and their representations ($\mathcal{V}$). Besides the labels $\mathcal{G}$ and $\mathcal{V}$, the statistical weights for the gapless critical models are trigonometric functions of two parameters θ (spectral parameter) and γ (anisotropy parameter). γ is related with the quantum group variable q through $q = e^{i\gamma}$. In addition, it is possible to introduce in the weights a set of discrete parameters $\epsilon_a = \pm 1$ $(1 \le a < \dim \mathcal{V})$ respecting the integrability. This has been done for $\mathcal{G} = A_n$ in Ref. [3] and can also be done for other Lie algebras.

We present in this Letter the exact solution for the A_n case (Perk-Schultz model) for all choices of $\epsilon_a = \pm 1$. The Perk-Schultz model is a vertex model in a square lattice where the links can be in $n+1$ different states. The nonzero statistical weights are given by (trigonometric regime)

$$\begin{aligned} R^{aa}_{aa}(\theta) &= \sin(\gamma + \epsilon_a \theta)/\sin\gamma , \\ R^{ab}_{ba}(\theta) &= G_{ab} \sin(\theta)/\sin\gamma, \quad a \neq b , \\ R^{ab}_{ab}(\theta) &= e^{i\theta \operatorname{sgn}(a-b)}, \quad a \neq b . \end{aligned} \tag{1}$$

Here $\epsilon_a = \pm 1$ and $G_{ab} G_{ba}^{-1} = 1$ (no sum on a,b).

This multicolor vertex model (number of colors is $n+1$) favors ferroelectric configurations (all links of the same color) for the colors a having $\epsilon_a = +1$. (We assume $|\theta| < \gamma/2$.) For colors b with $\epsilon_b = -1$, the weights (1) make alternating colored configurations more probable. Notice that reversing the signs of all ϵ_a $(1 \le a \le n+1)$ is equivalent to changing γ into $\pi - \gamma$. In spite of the fact that some weights are complex here, all results turn out to be physically meaningful. All weights are real in the hyperbolic regime that follows from (1) by $\theta \to i\theta$, $\gamma \to i\gamma$.

The eigenvalues and eigenvectors of the row-to-row transfer matrix for N sites associated with (1) can be written as [4]

$$\Lambda(\theta, \tilde{\lambda}) = \sum_{j=1}^{n+1} \Lambda^{(j)}(\theta, \tilde{\lambda}) , \tag{2}$$

where the first term

$$\Lambda^{(1)}(\theta,\tilde{\lambda}) = \prod_{j=2}^{n+1} [G_{j1}]^{p_{j-1}-p_j} \left(\frac{\sin(\gamma+\epsilon_1\theta)}{\sin\gamma} \right)^N \prod_{\alpha=1}^{p_1} \epsilon_1 \frac{\sinh(\lambda^{(1)}_\alpha + i\theta - i\gamma\epsilon_1/2)}{\sinh(\lambda^{(1)}_\alpha + i\theta + i\gamma\epsilon_1/2)} \tag{3}$$

dominates in the thermodynamic limit ($N = \infty$). The numbers $\lambda^{(j)}_{a_j}$, $1 \le j \le n$, $1 \le a_j \le p_j$, with $N \equiv p_0 \ge p_1 \ge \cdots \ge p_n \ge p_{n+1} \equiv 0$, are solutions of the nested Bethe-ansatz equations (BAE)

$$\begin{aligned} \prod_{j=1}^{n-1} [G_{ij} G_{j,i+1}]^{N_j} [\epsilon_{i+1}]^{p_{i+1}} [\epsilon_i]^{p_i - 1} &= \prod_{\beta=1}^{p_j} \frac{\sinh[\lambda^{(j)}_\alpha - \lambda^{(j)}_\beta - i\gamma\epsilon_{j+1}]}{\sinh[\lambda^{(j)}_\alpha - \lambda^{(j)}_\beta + i\gamma\epsilon_j]} \\ &\times \prod_{\beta=1}^{p_{j+1}} \frac{\sinh(\lambda^{(j)}_\alpha - \lambda^{(j+1)}_\beta + i\gamma\epsilon_{j+1}/2)}{\sinh(\lambda^{(j)}_\alpha - \lambda^{(j+1)}_\beta - i\gamma\epsilon_{j+1}/2)} \prod_{\beta=1}^{p_{j-1}} \frac{\sinh(\lambda^{(j)}_\alpha - \lambda^{(j-1)}_\beta + i\gamma\epsilon_j/2)}{\sinh(\lambda^{(j)}_\alpha - \lambda^{(j-1)}_\beta - i\gamma\epsilon_j/2)} , \end{aligned} \tag{4}$$

for $1 \le j \le n$, $1 \le \alpha \le p_j$. Here $N_j \equiv p_{j-1} - p_j$.

These BAE reduce to those solved in Ref. [2] when $\epsilon_a = -1$ for $1 \le a \le n+1$. As one sees from (4) the factors G_{ab} have the meaning of external fields and we will assume $|G_{ab}| = 1$ in this paper. In that case they are equivalent to gauge transformations or twists on the boundary conditions [2,5]. The effect of ϵ_a is more dramatic. When $\epsilon_j = -\epsilon_{j+1}$ (for a given j), we see in Eq. (4) that the phase describing the interaction of pseudoparticles in the jth step between them ($\lambda^{(j)}_\beta$, $1 \le \beta \le p_j$) vanishes. The interaction between pseudoparticles in different steps is always present. The attractive or repulsive character of the interactions can be changed at will by choosing the ϵ_a appropriately.

Let us now solve Eq. (4) for $N \to \infty$. The number of roots tends to infinity in this limit and they become closer and closer with a $1/N$ spacing. For the ground state, we find that they have a fixed imaginary part,

$$\lambda_a^{(j)} = \mu_a^{(j)} + i\tfrac{1}{4}\pi(1+\epsilon_j)\,, \tag{5}$$

where $\mu_a^{(j)} \in \mathcal{R}$.

Taking logarithms of Eq. (4) we find

$$\delta_{\epsilon_j,\epsilon_{j+1}}\epsilon_j \sum_{\beta=1}^{p_j} \Phi(\mu_a^{(j)} - \mu_\beta^{(j)}, \gamma) - \epsilon_j \sum_{\beta=1}^{p_j-1} \Phi(\mu_a^{(j)} - \mu_\beta^{(j-1)} + i\tfrac{1}{4}\pi(\epsilon_j - \epsilon_{j-1}), \gamma/2)$$
$$- \epsilon_{j+1} \sum_{\beta=1}^{p_{j+1}} \Phi(\mu_a^{(j)} - \mu_\beta^{(j+1)} + i\tfrac{1}{4}\pi(\epsilon_j - \epsilon_{j+1}), \gamma/2) = 2\pi I_a^{(j)} - \beta_j\,, \tag{6}$$

where β_j stands for the logarithm of the left-hand side of Eq. (4), $1 \le j \le n$, $1 \le a \le p_j$, and

$$\Phi(\lambda, z) \equiv i \ln \frac{\sinh(\lambda + iz)}{\sinh(\lambda - iz)}\,, \quad \bar{\Phi}(\lambda, z) \equiv -\Phi(\lambda + i\pi/2, z)\,, \tag{7}$$

and $I_a^{(j)} \in Z + \frac{1}{2}$. For the ground state the $I_a^{(j)}$ form monotonous sequences

$$I_{a+1}^{(j)} - I_a^{(j)} = 1\,. \tag{8}$$

Define the densities

$$\rho_j(\mu_a^{(j)}) = \lim_{N\to\infty} \frac{1}{N(\mu_{a+1}^{(j)} - \mu_a^{(j)})}\,. \tag{9}$$

Using the usual procedure [2], Eq. (6) yields in the $N = \infty$ limit a set of linear integral equations for the densities $\rho_j(\mu)$. We have for the ground state, where $\sigma_j \equiv \rho_j$,

$$\sigma_j(\mu) - \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{d\mu'}{2\pi} K_{jk}(\mu - \mu')\sigma_k(\mu') = \frac{\delta_{j1}}{2\pi}\Phi'(\mu, \gamma/2)\,. \tag{10}$$

Here

$$K_{jk}(\mu) = \epsilon_j \delta_{jk} \delta_{\epsilon_j,\epsilon_{j+1}} \Phi'(\mu,\gamma) - \epsilon_j \delta_{j,k+1}[\delta_{\epsilon_j,\epsilon_{j-1}}\Phi'(\mu,\gamma/2) - \delta_{\epsilon_j,-\epsilon_{j-1}}\bar{\Phi}'(\mu,\gamma/2)]$$
$$- \epsilon_j \delta_{j,k-1}[\delta_{\epsilon_j,\epsilon_{j+1}}\Phi'(\mu,\gamma/2) - \delta_{\epsilon_j,-\epsilon_{j+1}}\bar{\Phi}'(\mu,\gamma/2)]\,. \tag{11}$$

Equation (10) can be easily solved by Fourier transformation,

$$\sigma_j(\mu) = \int_{-\infty}^{\infty} \hat{\sigma}_j(k) e^{i\mu k}\, dk\,. \tag{12}$$

We find

$$\hat{\sigma}_j(k) = \frac{\sinh\{\frac{1}{2}k[(n+1-j)\pi/2 + (\pi/2-\gamma)\sum_{l=j+1}^{n+1}\epsilon_l]\}}{\sinh\{\frac{1}{2}k[(n+1)\pi/2 + (\pi/2-\gamma)\sum_{l=1}^{n+1}\epsilon_l]\}}\,. \tag{13}$$

Using Eq. (13) we get for the Cartan weights in the ground state

$$S_j = 2p_j - p_{j-1} - p_{j+1} = \frac{\epsilon_{j+1} - \epsilon_j}{(n+1)(1-2\gamma/\pi)^{-1} + \sum_{l=1}^{n+1}\epsilon_l}\,. \tag{14}$$

It follows from Eq. (14) that $|S_l| < 1$ for $0 < \gamma < \pi$. Therefore, unless all ϵ_j are equal we find ferrielectric behavior. That is, the $|S_j|$ values are larger than in the antiferroelectric case ($S_j = 0$) and smaller than in the ferroelectric case ($S_j = \pm 1$). More precisely, S_j behaves ferrielectrically provided $\epsilon_j \ne \epsilon_{j+1}$. Otherwise, when $\epsilon_j = \epsilon_{j+1}$, S_j exhibits an antiferroelectric character.

The free energy follows from Eqs. (2), (3), and (13),

$$f(n,\theta,\gamma) = -\lim_{N\to\infty} \frac{1}{N} \ln\Lambda(\theta,\lambda)$$
$$= 2\int_0^{\infty} \frac{dx}{x} \frac{\sinh(2x\theta)}{\sinh(\pi x)} \frac{\sinh\{x[n\pi/2 + (\pi/2-\gamma)\sum_{k=2}^{n+1}\epsilon_k]\}}{\sinh\{x[(n+1)\pi/2 + (\pi/2-\gamma)\sum_{k=1}^{n+1}\epsilon_k]\}} \sinh\{x[\pi/2 - \epsilon_1(\pi/2-\gamma)]\}\,. \tag{15}$$

Excited states are obtained when holes appear in the sequences $I_a^{(j)}$ [see Eq. (8)] as

$$I_{a+1}^{(j)} - I_a^{(j)} = 1 + \delta_{a,a_h}\delta_{j,j_h}\,, \tag{16}$$

where $\theta_h^j \equiv \lambda_{a_h}^j$ is the hole position at the step j_h [2].

The root density for such a state is given by

$$\sigma_{j,j_h}(\mu)=\delta_{j,j_h}\delta(\mu-\theta_h^j)-R_{j,j_h}(\mu-\theta_h^j)\,, \tag{17}$$

where $R_{lk}(\mu)$ is the resolvent of Eq. (10). That is,

$$R=[1-K]^{-1}\,. \tag{18}$$

We find

$$\hat{R}_{ll'}(2x)=\sinh(\pi x)\frac{\sinh\{x[l_<\pi/2+(\pi/2-\gamma)\sum_{k=1}^{l_<}\epsilon_k]\}}{\sinh[x(\pi-\gamma)]\sinh(x\gamma)}\frac{\sinh\{x[(n+1-l_>)\pi/2+(\pi/2-\gamma)\sum_{k=l_>+1}^{n+1}\epsilon_k]\}}{\sinh\{x[(n+1)\pi/2+(\pi/2-\gamma)\sum_{k=1}^{n+1}\epsilon_k]\}}\,. \tag{19}$$

The transfer matrix eigenvalue corresponding to a hole at θ_h^j in step j has the form

$$\Lambda_{\rm exc}(\theta,\theta_h^{(j)})\underset{N\to\infty}{=}\exp[-Nf(n,\theta,\gamma)-ig_j(\theta+i\theta_h^{(j)},\gamma)]\,, \tag{20}$$

where we find

$$g_j(\phi,\gamma)=\Phi\left\{\kappa_n\phi,\frac{\pi}{4}\kappa_n\left[j+\left(1-2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j}\epsilon_k\right]\right\}-\frac{\pi}{2}\kappa_n\left[j+\left(1-2\frac{\gamma}{\pi}\right)\sum_{k=1}^{j}\epsilon_k\right]. \tag{21}$$

Here

$$\kappa_n\equiv\left[(n+1)/2+(\tfrac{1}{2}-\gamma/\pi)\sum_{k=1}^{n+1}\epsilon_k\right]^{-1}.$$

Notice that Eqs. (13)-(21) are invariant under

$$\epsilon_j\to-\epsilon_j,\quad \gamma\to\pi-\gamma\,,$$

as it must be. When $\epsilon_j=-1$ $(1\le j\le n+1)$ they reduce to the results in Ref. [2].

We see from (21) that the model is gapless in the present trigonometric regime since $g_j(-\infty,\gamma)=0$. [It has a gap in the hyperbolic regime following from Eq. (1) upon $\theta\to i\theta,\gamma\to i\gamma$.] We can then apply the light-cone approach [6] to derive a massive quantum field theory in an appropriate scaling limit. In this approach the energy E and momentum P of the excitations is given by

$$E\pm P=\lim_{a\to 0,\,i\theta\to\infty}\frac{g^{(j)}(\pm\theta+i\theta_h^{(j)},\gamma)}{a}\,. \tag{22}$$

We let $\theta\to i\infty$ and the lattice spacing $a\to 0$ such that

$$\mu=(1/a)\exp(-i\theta\kappa_n)\,, \tag{23}$$

where μ is a fixed mass unit. The energy-momentum dispersion law results in

$$E_l=m_l\cosh(\kappa_n\theta_h^{(l)}),\quad P_l=m_l\sinh(\kappa_n\theta_h^{(l)})\,,$$

where

$$m_l=\mu\sin\left\{\frac{\pi}{2}\kappa_n\left[l+\left(1-2\frac{\gamma}{\pi}\right)\sum_{k=1}^{l}\epsilon_k\right]\right\}. \tag{24}$$

That is, we find relativistic particles with the mass spectrum (24). We want to stress that this mass spectrum depends on the continuous parameter γ. This is in contrast with all mass spectra found up to now for integrable quantum field theories, which are all γ independent [1,2,6,7]. Indeed, the gap (sometimes called mass) is coupling dependent in models like the eight-vertex model. At the quantum field theory level γ stands for a coupling constant (or a function of it).

The S matrix between a hole at branch l and another at l' follows from Eq. (19) applying the method of Ref. [8] (that is, the S matrix between a particle m_l and a particle $m_{l'}$). It reads $S_{ll'}(\phi)=\exp[i\delta_{ll'}(\phi)]$, where ϕ is the relativistic rapidity and

$$\delta_{ll'}(\phi)=2\pi\int_0^{\phi/\kappa_n}\sigma_{ll'}(\lambda)d\lambda\,. \tag{25}$$

A look at Eqs. (19) and (26) shows that this S matrix can be expressed as an infinite product of Γ functions.

Besides this scaling limit yielding an integrable massive quantum field theory, we can take the trivial continuous limit, leading to a conformal field theory.

Let us sketch the derivatives of the finite-size corrections and give the results for the conformal properties of this continuous limit. The finite-size corrections to the free energy can be expressed analogously to Ref. [2] as

$$\begin{aligned}L_N(n,\theta,\gamma)&=f_N(n,\theta,\gamma)-f(n,\theta,\gamma)\\&=-\sum_{l=1}^{n}\left[\int_{-\infty}^{-\Lambda_l^-}+\int_{\Lambda_l^+}^{\infty}\right]d\lambda_l f_l(\lambda_l)\sigma_N^{(l)}(\lambda_l)\\&\quad+\frac{1}{2N}\sum_{l=1}^{n}[f_l(\Lambda_l^-)+f_l(\Lambda_l^+)]+\frac{1}{12N^2}\sum_{l=1}^{n}\left[\frac{f_l'(\Lambda_l^+)}{\sigma_N^{(l)}(\Lambda_l^+)}-\frac{f_l'(\Lambda_l^-)}{\sigma_N^{(l)}(\Lambda_l^-)}\right]+(\text{higher-order terms})\,.\end{aligned} \tag{26}$$

Here $\pm\Lambda_l^{\pm}$ are the largest positive and negative roots of the BAE (4) in the lth branch. The Fourier transforms

$$X_l^{\pm}(w)=\int_{-\infty}^{\infty}\exp(iwt)\theta(\pm t)\sigma_N^{(l)}(\Lambda_l^{+}+t)dt \tag{27}$$

are solutions of the matrix Riemann-Hilbert problem

$$X_k^{-}(x)+\sum_{l=1}^{n}\hat{R}_{lk}(w)X_l^{+}(w)=e^{-iw\Lambda_k^{+}}\hat{\sigma}_k(w)+\frac{1}{2N}\left(-1+\sum_{l=1}^{n}\hat{R}_{lk}(w)\right)-\frac{iw}{12N^2}\sum_{l=1}^{n}(\delta_{kl}-1)\frac{\hat{R}_{lk}(w)}{\sigma_N^{(l)}(\Lambda^{+})}\,, \tag{28}$$

where we used $\Lambda_l^{+}=\Lambda_l^{-}=\Lambda$.

The solution of this problem is analogous to Refs. [2] and [7]. We find

$$L_N(n,\theta,\gamma)=-\frac{\pi n}{6N^2}\sin(\kappa\theta)-\frac{2\pi i}{N^2}[\Delta e^{-i\kappa\theta}-\bar{\Delta}e^{i\kappa\theta}]\,, \tag{29}$$

where

$$\Delta=\frac{1}{8}\sum_{l',l=1}^{n}\left[2h_l^{+}+\frac{\gamma}{2\pi}(\epsilon_{l+1}+\epsilon_l)S^{l}\right]\hat{R}_{ll'}(0)\left[2h_{l'}^{+}+\frac{\gamma}{2\pi}(\epsilon_{l'+1}+\epsilon_{l'})S^{l'}\right]. \tag{30}$$

$\hat{R}_{ll'}(0)$ follows From Eq. (19) and

$$\hat{R}_{lj}^{-1}(0)=-2\delta_{lj}\left(\frac{\epsilon_{l+1}+\epsilon_l-2}{4}-\frac{(\epsilon_{l+1}+\epsilon_l)\gamma}{2\pi}\right)+\delta_{l,j+1}\left(\frac{\epsilon_{l-1}+\epsilon_l}{2}-\frac{(\epsilon_l)\gamma}{\pi}\right)+\delta_{l,j-1}\left(\frac{\epsilon_{l+1}+\epsilon_l}{2}-\frac{(\epsilon_{l+1})\gamma}{\pi}\right) \tag{31}$$

and $S^{(l)}$ stands for the Cartan weights of the state,

$$S^{(l)}=2p_l-p_{l+1}-p_{l-1}\,. \tag{32}$$

$\bar{\Delta}$ follows from Δ by exchanging $h_l^{+}\leftrightarrow h_l^{-}$. We denote by $h_l^{\pm}$ the number of holes at step l and beyond $\pm\Lambda$. Since the speed of sound here is $v=\sin\kappa\theta$, Eqs. (29) and (30) tell us that the central charge is $c=n$ for the Perk-Schultz model.

For $n=1$, $\epsilon_1=-1$, and $\epsilon_2=+1$ the Perk-Schultz model becomes a solvable six-vertex model. This turns out to be a free model since the BAE become in this case a set of p_1 decoupled equations. This six-vertex model possess ferrielectric behavior since the ground-state spin equals

$$S=2\gamma/\pi-1 \tag{33}$$

[from Eq. (14)]. The free energy follows from Eq. (15):

$$f(\theta,\gamma)=2\int_0^{\infty}\frac{dx}{x}\sinh(2x\theta)\left[\frac{\sinh[x(\pi-\gamma)]}{\sinh(\pi x)}\right]^2. \tag{34}$$

It differs from the usual six-vertex model [2], showing that the two models are inequivalent. In the massive scaling limit [Eq. (23)], we find (for $n=1$, $\epsilon_1=-1$, and $\epsilon_2=+1$) a single free particle with mass $\mu\sin\gamma$. Notice that the S matrix (26) equals unity in this case as one would expect from the BAE decoupling.

E.L. was supported in part by CAPES (Brazil). LPTHE is unité associée au CNRS No. 280.

[1] *Yang-Baxter Equation, Conformal Invariance and Integrability,* edited by M. N. Barber and P. A. Pearce (World Scientific, Singapore, 1989); *Integrable Systems in Quantum Field Theory and Statistical Mechanics,* edited by M. Jimbo *et al.*, Advanced Studies in Pure Mathematics Vol. 19 (Academic, New York, 1990).

[2] H. J. de Vega, Int. J. Mod. Phys. A **4**, 10 2371 (1989); Nucl. Phys. B (Proc. Suppl.), **18A**, 229 (1990).

[3] J. H. H. Perk and C. L. Schultz, in *Nonlinear Integrable Systems—Classical Theory and Quantum Theory,* edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983), p. 135; C. L. Schultz, Phys. Rev. Lett. **46**, 629 (1981); J. H. H. Perk and C. L. Schultz, Phys. Lett. **84A**, 407 (1981).

[4] C. L. Schultz, Physica (Amsterdam) **122A**, 71 (1983); E. Lopes, LPTHE Report No. 91/30 (to be published).

[5] H. J. de Vega and E. Lopes, Phys. Lett. B **186**, 180 (1987).

[6] C. Destri and H. J. de Vega, Nucl. Phys. **B290**, 363 (1987); J. Phys. A **22**, 1329 (1989).

[7] H. J. de Vega and E. Lopes, LPTHE Report No. 90/42 (to be published).

[8] V. E. Korepin, Theor. Math. Phys. **41**, 953 (1981); N. Andrei and C. Destri, Nucl. Phys. **B231**, 445 (1984).