Selection of Saffman-Taylor Fingers in the Sector Geometry

R. Combescot and M. Ben Amar
Laboratoire de Physique Statistique, Ecole Normale Superieure et Universite Pierre et Marie Curie,
24 rue Lhomond, F-75231 Paris CEDEX 05, France
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We study numerically and analytically the selection of Saffman-Taylor fingers in cells having the shape of a disk sector. For divergent fingers we find results which are qualitatively new compared to the standard linear geometry. Instead of a discrete set of solutions, all converging to a relative width of $\frac{1}{3}$ for zero surface tension, we find that the fingers disappear below some surface tension. This happens by the merging of neighboring branches of the discrete set of solutions. This behavior is in (quantitative) agreement with experimental results.

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Despite all the recent progress in the field [1], the Saffman-Taylor instability continues to attract a large interest. The reason is the wide variety of phenomena displayed in contrast with the apparent simplicity of the physical situation. The Saffman-Taylor (ST) instability is perhaps the simplest example of pattern selection. In addition to standard fingers, it displays fractal structures [2] and even dendritic growth [3] under anisotropic conditions. These are remarkably similar to a diffusion-limited aggregate, which is itself a standard model of crystal growth.

The standard finger-width selection is by now well understood in the linear geometry [4]. On the other hand, the circular geometry is much less under control. Thomé et al. [5] have devised a clever experiment to bridge between the two geometries by growing Saffman-Taylor fingers in Hele-Shaw cells with the shape of a disk sector. They observed fingerlike interfaces which at large velocities displayed a self-similar growth. This is a nice generalization of the finger moving at constant velocity found in the linear case. The minimum angular width of these fingers (found at large velocities) varies roughly linear with the sector angle $2\theta_0$. They observed an increasing sensitivity to tip splitting for larger angle which made it impossible to observe the asymptotic regime of large velocities beyond roughly $2\theta_0 \approx 20^\circ$. This phenomenon has already been used by Sakar [6] in a theory of fractal growth in radial geometry.

In this paper we consider theoretically the width selection of Saffman-Taylor fingers in this sector geometry. This problem has already be considered by Brener et al. [7] for the $2\theta_0 = 90^\circ$ sector. We will concentrate on the divergent case since the convergent situation happens to be quite similar to the linear geometry. In the divergent case we find a qualitatively new phenomenon. In the linear geometry one has a set of discrete solutions all converging to a relative width $\lambda = \frac{1}{3}$ for vanishing surface tension $T$. Here instead we find that the solutions disappear below some surface tension (or equivalently above some sector angle) by a merging of the solutions corresponding to neighboring branches [8]. This disappearance of the selected solution provides a natural and intrinsic explanation for the instability observed experimentally, with a rough quantitative agreement (experiment is difficult).

For a general sector angle $2\theta_0$ and surface tension $T$, our results have been obtained numerically by solving the problem on the finger interface. Because of branch merging, the spectrum for the finger width displays curled branches (see Fig. 1). In order to understand the physical origin of this new behavior, we have for small angle and surface tension performed an analytical study by a complex extension. The resulting differential equation (inner problem) displays indeed the branch-merging phenomenon. The reason is found by a WKB analysis. Let us recall that, for the linear geometry, one finds either a single singularity on the channel axis for $\lambda < \frac{1}{2}$ or two singularities on the channel sides for $\lambda > \frac{1}{2}$. Here instead we find always three singularities (for the divergent case). One of them is axislike, and the other two are sidelike. When these latter ones dominate (are “closer”

![FIG. 1. Finger-width spectrum $R$ as a function of $D^{-1}$. Squares: numerical results on the finger (with the axis $D^{-1}$ upscaled by a factor of 1.78). Crosses: experimental results (also upscaled). Open squares: numerical integration of Eq. (6). Solid interrupted curves: WKB results. Dashed curves: simple approximation to WKB. Horizontal line: WKB limit. Without branch merging the branches would roughly keep following the solid curves of the WKB results.](Image)
to the finger), the situation is analogous to the linear case with a set of discrete branches. However, at low $T$ the axis singularity always becomes dominant and, as for the linear channel, this leads to a disappearance of the solutions, which is done effectively by branch merging. The location of this disappearance in the WKB analysis (and all the width spectra) is in very good agreement with the direct solution of the inner problem, which agrees very well itself for $2\theta_0 = 20^\circ$ with the numerical results on the finger (Fig. 1). Actually with a proper adjustment, the scaling found in the inner problem for small angle agrees with the numerics up to $2\theta_0 = 70^\circ$.

We take the disk sector symmetrical with respect to the positive $x$ axis, the apex being at the origin. We look for a complex potential $w = \phi + i\psi$, an analytical function of the complex position $z = x + iy$ in the viscous fluid region, with complex velocity $u = u_q - i\omega_z = dw/dz$, being at infinity $u = 1/\theta_2 z$, with $\arg(u) = \pm \theta_0$ on the cell sides. On the finger the components, normal to the interface, of the fluid velocity and of the finger velocity must be equal (continuity equation), and $\phi = a/R$, where $R$ is the radius of curvature of the interface and $a = Th^2/12\mu$, with $b$ the cell thickness and $\mu$ the fluid viscosity. The finger angular width $2\theta_1$ is the finger angle at the origin and the relative width is $\lambda = \theta_1/\theta_0$. Self-similar fingers are possible [5] only if they grow as $(1 + 3A t)^{1/2}$, where $A$ is a constant which plays the role of the finger velocity. We take $t = 0$ in the following.

The numerical scheme rests on the hodograph method of Mclean and Saffman [9] (McS) extended to the present situation. We recover the McS linear geometry by the conformal mapping $\theta_0 Z = \text{In} z$ which transforms the wedge into a linear cell. The major problem comes from the continuity equation. In order to recover the McS formalism, we build an analytical function $H(Z)$. Its imaginary part on the finger, after derivation, plays the same role as $U_0$ in McS. We introduce a generalized complex potential $W = \lambda [w(Z) - H(Z)]/Q_0 (1 - \lambda)$, which vanishes on the finger and the axis, and is equal to $-1$ on the upper wall ($2Q_0$ is the fluid flux through the cell). By $\Sigma = e^{-iW}$ we transform the upper fluid-flow space into the upper half plane $\Sigma = s + i\tau$. In the $\Sigma$ plane, the upper wall is on $[-\infty, 0]$, the finger on $[0, 1]$, and the cell axis on $[1, \infty]$. In this plane, $\phi = a/R$ gives a relation between the curvature and the derivative of the potential $dW/d\Sigma = -q e^{-i(1 - \lambda)}$, which is merely the complex velocity in the traditional ST experiment. Here, however, this physical meaning is lost. In terms of $s$ this equation reads

$$\frac{\kappa q^2 q}{d_s} \frac{\partial}{\partial s} \exp(-q(1 - \lambda)) \left(\frac{q}{s} \frac{\partial q}{\partial s} + \frac{\theta_0}{\pi} (1 - \lambda) \sin \tau \right) = -\frac{q}{1 - \lambda} \frac{q}{\pi q_0} \int_0^t dt \exp(2q(1 - \lambda)) \sin \tau \frac{q(t - s)}{q(t - s)},$$

where $\kappa = \pi a(1 - \lambda) Q_0 (1 - \lambda)^2 R_0^{1/2}$ ($R_0$ is the finger length), with the boundary conditions $\tau(0) = 0$, $\tau(1) = -\pi/2$, $q(0) = 1$, and $q(1) = 0$. For $\theta_0 = 0$, one recovers the McS equations. Two other equations have already been given by McS: We notice that $q e^{-i(1 - \lambda)}$ is an analytical function of $Z$, which gives from Cauchy's theorem

$$\ln q = -\frac{s}{n} \int_0^t dt \frac{\tau(t)}{t(t - s)}.$$  

Here we have used the fact that $\tau(s)$ vanishes everywhere on the real $s$ axis, except for $s \in [0, 1]$. The other equation simply indicates that $Z$ is an analytical function of $W$:

$$X(s) + iY(s) = -\frac{1 - \lambda}{\pi} \int_0^t dt \frac{\exp(i\tau)}{tq}.$$  

For the sector geometry, one has to solve these three coupled equations for $q$, $\tau$, and $x$ instead of two. Nevertheless, our algorithm, which simply reproduces the technical aspects [10] published in Ref. [9] is especially effective. Using always the same starting function, that is, the ST solutions at zero surface tension in the linear geometry, it is able to select several eigenvalues $\lambda_n$ for any angle $\theta_0$ and any $\kappa$ value (less than 10000). Since it is very fast, we are able to explore an extensive domain of parameters both in the convergent and in the divergent flow regime [11]. However, we focus here on the domain which can be easily understood in terms of our analytical analysis.
We turn now to the analytical investigation [11] of selection in the limit of low surface tension $T$. The method can be applied for any angle [11], more so since we have an explicit analytical solution [12] for these fingers at $T=0$. However, in order to obtain explicit results, one ends up with numerical calculations. Therefore we will concentrate on sectors with small angles where essentially the theory can be carried out completely explicitly. We will see that this limit displays all the behaviors found numerically for a general angle, and that the physical reasons for it will appear quite clearly and simply; this limit provides all the qualitative understanding in which we are interested.

The first step is to have at $T=0$ an analytical solution for the continuum of self-similar fingers. For small angle it can be shown [11] to be, for proper time and length scale,

$$\ln z - w \theta_0 = (2\theta_0/\pi)(1 - \lambda) \ln (1 + e^{-\eta})$$  \hspace{1cm} (4)

which is the first term of a $\theta_0$ expansion. This solution is in good agreement with the small-$\theta_0$ experimental results [5] (the finger shape is obtained by setting $\phi=0$ in the complex potential).

By taking the derivative, with respect to arclength along the finger, of the boundary condition $\phi = 2\pi R$ and combining with the relation between finger and fluid velocity, we obtain [13] a differential equation along the finger:

$$\frac{d^2f}{dz^2} + \frac{2\lambda \theta_0 - z}{f} = 0$$  \hspace{1cm} (5)

where $\epsilon = 2\alpha /A = 2\alpha \theta_0$; the complex velocity $u$ can be obtained from Eq. (4) to lowest order in surface tension. Here $f = e^{i\theta}$ with $\theta$ being the angle between the normal to the finger and the cell axis. In Eq. (5), the first term acts as a singular perturbation and the selection condition is obtained by requiring that no transcendental divergent term is produced by the perturbation.

This condition is most conveniently expressed by performing an analytical continuation of Eq. (5) in order to go the singular points which generate the divergence. In the present case, this continuation merely amounts to extending Eq. (5) into the viscous fluid [14]. Here $z^*$ must be understood as the analytical continuation of $z^*(\omega)$ or $z^*(w)$. From this differential equation a perturbation expansion in $\epsilon$ can be generated for $f$ around the $\epsilon=0$ solution $f_0 = (2\lambda \theta_0 \omega /z - z^*/z)^{-1/2}$. However, this expansion breaks down around the singularities of $f_0$, where $\epsilon d^2f_0 / dz^2$ diverges. These singular points are the zeros of the right-hand side of Eq. (5). For small $\theta_0$ and $\lambda \approx \frac{1}{2}$ (which is expected for small $T$ by continuity from the $\theta_0=0$ case), these singularities are found for $z \approx 1$: They are located near the finger tip compared to the finger length. However, they satisfy $\epsilon e^{\eta w} \gg 1$ and are not located near the finger tip compared to the channel width $2\theta_0$. For this range $2\lambda \theta_0 \omega \approx 1 + 2\eta + e^{-\eta w}$, with $\eta = \lambda - \frac{1}{2}$, and $z \approx z^* \approx 1 + \theta_0 \omega$. It is then more convenient to use the rescaled variable $y = \sigma^{1/3} e^{-\eta w/2}$ and the function $F = \sigma^{1/3} f$, with $\sigma = \epsilon (\pi / 2\theta_0)^2$. This leads to the equation

$$y \frac{d}{dy} \left[ y \frac{dF}{dy} \right] + \frac{1}{F^2} = y^2 + C + D \ln y$$  \hspace{1cm} (6)

with (as a definition for the nonlinear eigenvalue $C$)

$$\frac{\eta}{2\theta_0} - \frac{1}{2} \ln \frac{\pi}{2\theta_0} = \frac{C}{D} + \frac{1}{2} \ln D = R, \quad D = \frac{2\theta_0}{\pi} \sigma^{-2/3}$$  \hspace{1cm} (7)

This nonlinear equation has to be solved with the boundary condition $F \approx 1/y$ for large $|y|$ and $|arg y| \leq \pi/2$. For $D \to 0$, i.e., $\theta_0 \to 0$ or large $\sigma$, the equation reduces to the one studied for the linear case [15]. We note that Eq. (7) provides a mapping between the problems for different surface tension and angle, but same $D$. This scaling law is a generalization of the relation $\eta \sim e^{2/3}$ for linear fingers.

Equation (6) can be solved numerically. One finds for $C(D)$ discrete branches, shown in Fig. 2, which display for large $D$ the branch-merging phenomenon. When translated for the finger width as a function of $D^{-1}$, one finds the results shown in Fig. 1 which agree with numerical results on the finger. In particular, the minimum width for $2\theta_0=20^\circ$ is $\lambda = 0.63$ which agrees with these results within numerical uncertainty. In order to gain more insight into branch merging, we proceed to a WKB treatment of Eq. (6). Although valid in principle only for large $C$, this has proved to be quite successful qualitatively and quantitatively for linear fingers [15]. This is done by the new scaling $x = y r$ and $F = r G$, which gives

$$y^2 x \frac{d}{dx} \left[ x \frac{dG}{dx} \right] + \frac{1}{G^2} = x^2 + 1 + \delta \ln x$$  \hspace{1cm} (8)

with $\gamma^2 = \delta /D$ and $R = 1/\delta + \frac{1}{2} \ln \delta$. For small $\gamma$, Eq. (8) has the solution $G_0 = (x^2 + 1 + \delta \ln x)^{-1/2}$ to lowest order.

The small corrections $h$ to this solution are obtained by linearizing Eq. (8) with $G = G_0 + h$. The possible transcendental terms generated satisfy the homogeneous part of the linearized equation. A WKB treatment gives

$$h \sim G_0^{3/4} \exp \left[ \pm \left( 2 \gamma - 1 \right)^{1/2} \int_0^{x_0} dx G_0^{-3/2}/x \right].$$

But this approximation is not valid near the zeros of $G_0^{-1}$, i.e., of the right-hand side of Eq. (8) (these are the singularities of $f_0$).

The equation $x^2 + 1 + \delta \ln x \approx 0$ always has three solutions. One of them $x_0$ is real with $x_0 > 0$ and is analogous to the axis singularity found for the linear finger [15]. The two other roots $x_\pm$ are complex conjugate with $\pi/2 < \arg x_\pm < \pi$ and are analogous to the two side singularities of the linear case [15]. Therefore, instead of having either one axis or two side singularities as in the linear case, all three are present in the sector geometry. Whenever the axis singularity $x_0$ “dominates,” i.e., is “nearer” to the finger, we find the same situation as for the linear finger with an axis singularity and there is no solution. When the side singularities $x_\pm$ dominate, we find solutions as for the linear case. The transition between the two regimes occurs when $x_0$ and $x_\pm$ are on the same Stokes line,

$$\Re \int_0^{x_0} dx G_0^{-3/2} = 0,$$

which gives the limiting values $\delta_0 = 0.55$ and $R_0 = 1.52$ for $\delta$ and $R$. The corresponding boundary in the $(C,D)$ plane coincides very well with the place [16] where branch merging occurs as seen in Fig. 2. Numerical integration of Eq. (6) leads to an effective $R_0 = 1.25$ for the first “curl” and $R_0 = 1.4$ for the second one. The corresponding minimum width is given by

$$\eta_{\text{min}} = \frac{\pi}{\theta_0} |R_0 + \frac{1}{2} \ln (\pi/2 \theta_0)|.$$  \hspace{1cm} (10)

This result shows that $\eta_{\text{min}}$ has a weak singularity for $\theta_0 = 0$, but the overall behavior is roughly linear, in agreement with experiment. Note, on the other hand, that $\eta$ is linear with $\theta_0$ at constant $c$, since $C$ is essentially constant on the lower branch.

In the regime $\delta \leq \delta_0$, we require that $h$ is exponentially small for large $|x|$, $|\arg x| \leq \pi/2$. The situation is quite analogous to the linear case [4,15] and leads in the same way to the selection condition

$$I(\delta) = \frac{2}{\pi} \Im \int_0^{x_0} dx G_0^{-3/2} = (2\gamma - 1)^{1/2}(n + \frac{3}{4}).$$  \hspace{1cm} (11)

Actually we have neglected here a very small correction due to the fact that $x_\pm$ are no longer purely imaginary as in the linear case [11,13]. Once $\delta$ is found from Eq. (11), the finger width is obtained from $2n\pi/\theta_0 = 2\delta + \ln (\pi \delta/2 \theta_0)$. For $\delta = 0$, $I(0) = 1$ and we recover the result for the linear finger. When $I(\delta)$ is calculated numerically, one obtains the results shown in Fig. 2, which agree very well with the numerical integration, except naturally near branch merging. Since $\delta$ is never large, it is possible to expand $I(\delta)$ as $I(\delta) - 1 \approx (3\delta/8)(3\ln 2 - \pi/2) = 0.196$. This gives a very good approximation to the WKB result as seen in Fig. 2. The corresponding result for the finger width is

$$\eta = C_0 D^{-1} - \frac{1}{2} \ln \frac{2 \theta_0 C_0 D^{-1}}{\pi} - 0.25,$$  \hspace{1cm} (12)

with $[15] C_0 = 2^{3/3}(n + \frac{3}{4})^{3/3}$ and $n = 0, 1, \ldots$. Together with the WKB limit Eq. (10), it gives a quite good qualitative and quantitative description of the finger-width picture (see Fig. 1).

In conclusion, our WKB analysis demonstrates clearly that branch merging occurs physically when the axis singularity is no longer hidden by the side singularities and becomes dominant. Then the finger disappears exactly as it does in the linear case for $\lambda < \frac{1}{2}$. Let us point out that this axis singularity is a physical object which can be manipulated experimentally, as shown recently [14]. Therefore we can safely predict that a bubble or a disk in front of the finger will produce narrow fingers as for the linear case.

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[8] A similar merging phenomenon has been found very recently in solidification in a channel by D. A. Kessler and H. Levine (private communication).
[16] Obtaining the actual merging is beyond the present WKB approach, since one should describe precisely how the axis singularity takes over the side singularities.