# Surface Waves in Nonsquare Containers with Square Symmetry

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Experiment and theory for surface waves in square containers suggest that the effective symmetry may be larger than the geometric symmetry of the container cross section. The extra symmetries complicate the construction of normal forms and appear to stabilize effects in the experments that would otherwise be nongeneric. These symmetries may be directly observed if the sidewalls are deformed to a nonsquare cross section that retains square symmetry. The effects of such a deformation are analyzed.

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Experiments on parametrically driven waves provide an interesting context for bifurcation and pattern formation in symmetric systems. Examples recently considered include standing-wave convection in binary mixtures [1], oscillatory "Williams rolls" in nematic liquid crystals [1], and surface waves in fluids. In particular, surface waves excited on a vertically oscillated fluid layer have been extensively studied as a very rich nonlinear system revealing various coherent patterns, including standing waves, precessing modes, and localized solitonlike states, as well as low-dimensional chaotic dynamics and a transition to spatiotemporal chaos [2,3].

It is now well understood that the qualitative features of a pattern-forming bifurcation are largely shaped by the type of instability and symmetries of the system and basic state [4]. Consequently, there is considerable theoretical interest in identifying the relevant group of symmetries of a given experiment and formulating suitable normal-form equations for the critical modes. The surface-wave experiments provide an especially interesting setting for testing the predictions of these symmetric normal forms particularly when the container dimensions are comparable to the wavelengths of the standing-wave pattern. In this low-aspect-ratio regime, the container cross section $\Omega$ defines a geometric symmetry $\Gamma$ that must be appropriately included in the normal-form symmetry. However, in both circular and square geometries, theories that model the experimental observations via normal forms based on $\Gamma$ alone have not been entirely successful; either certain qualitative features of the observed bifurcations were absent or else nongeneric assumptions were required on the nonlinear terms [5]. These discrepancies suggest there are additional qualitative features of the experiments that must be accounted for in the structure of the normal form. For example, if we regard the experiments as a weakly dissipative system then special features of the underlying conservative dynamics may be important [6]. Indeed recent work has shown that normal forms incorporating a weakly imperfect time-reversal symmetry naturally exhibit the near degeneracy previously introduced by Silber and Krupa on an *ad hoc* basis [7].

In this paper, I consider a further symmetry, in addition to time reversal and $\Gamma$, that appears to explain several puzzling features of the waves in square geometry [8]. Physically this symmetry is a remnant of the horizontal translation symmetry of an infinite fluid layer (no sidewalls), but its conjectured role in the experiment of Simonelli and Gollub (SG) is a subtle consequence of the square container cross section and the relative weakness of both viscous and capillary (surface tension) effects. This additional symmetry cannot appear in circular geometry and its impact on waves in rectangular geometry is less important. Unlike time reversal, even when this symmetry is mathematically exact it is a true symmetry only of the center manifold dynamics and does not belong to the symmetry group of the fluid model, i.e., hydrodynamics plus boundary conditions [9]; that is, we have an additional symmetry in the dynamics of the critical modes that is not shared by the full theory. After discussing the evidence for this effect, I propose an extension of the experiment of SG to test the theory; namely, to study the bifurcation of standing-wave patterns in a square container as the sidewalls are progressively distorted to form a nonsquare cross section that retains square symmetry [10].

For an incompressible, inviscid, irrotational fluid layer oscillating vertically with a displacement $A\cos 2\pi f_0 t$, the free-boundary problem was formulated by Benjamin and Ursell in terms of a velocity potential $\mathbf{u}(x,y,z,t)=\nabla\phi$ and a deformation field for the free surface $z=\zeta(x,y,t)$ [11]. Incompressibility $\nabla\cdot\mathbf{u}=0$ combined with the boundary condition on the normal velocity $\hat{\mathbf{n}}\cdot\mathbf{u}=0$ at the sidewalls and container bottom $(z=-h)$ require $\nabla^2\phi=0$ and $\hat{\mathbf{n}}\cdot\nabla\phi=0$ (sidewalls and bottom) for the potential. A Neumann boundary condition (NBC) provides the simplest compatible boundary condition on the free surface [12],

$$\hat{\mathbf{n}}\cdot\nabla\zeta=0 \text{ (sidewalls)}, \tag{1}$$

plus a kinematic boundary condition for $\zeta$ on the interior of $\Omega$. The boundary-value problem for $\phi$ is solved by $\phi(x,y,z,t)=\sum_{mn}b_{mn}(t)\psi_{mn}(x,y)H_{mn}(z)$, where $H_{mn}(z)=(\kappa_{mn}\sinh\kappa_{mn}h)^{-1}\cosh\kappa_{mn}(h+z)$ and $\psi_{mn}$ satisfies

$$\nabla_\perp^2\psi_{mn}+\kappa_{mn}^2\psi_{mn}=0,\quad \hat{\mathbf{n}}\cdot\nabla_\perp\psi_{mn}=0 \text{ (sidewalls)}. \tag{2}$$

By virtue of (1) and (2), $\zeta$ has an expansion $\zeta(x,y,t)=\sum_{mn}a_{mn}(t)\psi_{mn}(x,y)$ whose amplitudes satisfy the Mathieu equation in linear approximation ($\tau\equiv 2\pi f_0 t$):

$$\frac{d^2a_{mn}}{d\tau^2}+(\alpha_{mn}-\beta_{mn}\cos\tau)a_{mn}=0, \tag{3}$$

where $\alpha_{mn}=(\omega_{mn}/2\pi f_0)^2$ and $\beta_{mn}=A\kappa_{mn}\tanh\kappa_{mn}h$. The wave frequencies satisfy $\omega^2_{mn}=\kappa_{mn}\tanh(\kappa_{mn}h)[g+\delta\kappa^2_{mn}/\rho]$, where $g$ is the gravitational acceleration, $\delta$ the surface tension, and $\rho$ the density [11].

If we take the cross section $\Omega$ to be a square of length $\pi$ on a side (Fig. 1), the geometric symmetry $\Gamma$ is generated by reflection about $x=\pi/2$ and reflection across the diagonal, $\gamma_1:(x,y)\rightarrow(\pi-x,y)$ and $\gamma_2:(x,y)\rightarrow(y,x)$, and the normalized eigenfunctions (2) are $\psi_{mn}(x,y)=(2/\pi)\cos mx\cos ny$. For $n\neq m$, the eigenvalue $\kappa^2_{mn}=m^2+n^2$ is degenerate with two symmetry-related eigenfunctions (*pure modes*): $\psi_{nm}=\gamma_2\cdot\psi_{mn}$. However, the *mixed modes*, $\psi^{\pm}_{mn}\equiv\psi_{mn}\pm\psi_{nm}$, provide a more natural basis set for the eigenspace; these states are characterized by a diagonal reflection symmetry $\gamma_2\cdot\psi^{\pm}_{mn}=\pm\psi^{\pm}_{mn}$. Under reflection in $x$, the transformation properties of $\psi^{\pm}_{mn}$ depend on the even-odd parity of $m+n$:

$$\gamma_1\cdot\psi^{\pm}_{mn}=(-1)^m\times\begin{cases}\psi^{\pm}_{mn}, & m+n \text{ even},\\ \psi^{\mp}_{mn}, & m+n \text{ odd}.\end{cases} \tag{4}$$

For each $(m,n)$ there is an infinite set of instability zones in the $(A,f_0)$ parameter space; in the presence of dissipation, the instability corresponding to subharmonic oscillations has the lowest threshold. Consequently, this is usually the first instability observed experimentally, leading to standing waves at frequency $f_0/2$.

The parameter space of SG is shown in Fig. 2 indicating some of the mode numbers of the observed standing waves; a given transition is referred to below as even (odd) parity if $m+n$ is even (odd). The relevant aspects of the bifurcation to standing waves can be most simply described by considering transitions found at frequencies above resonance indicated by the arrows in Fig. 2. In each case a continuous transition to a supercritical pure mode is observed (either $\psi_{mn}$ or $\psi_{nm}$ depending on initial conditions); furthermore, an experimental reconstruction of the phase space reveals additional unstable mixed-mode states. The bifurcation diagram is shown in Fig. 2, diagram (a).

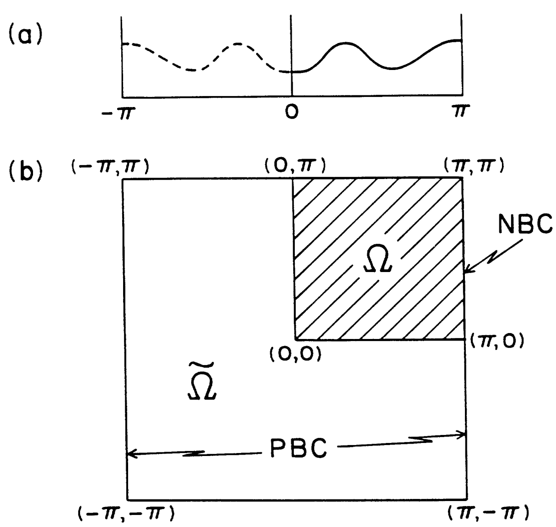


FIG. 1. (a) The extension by reflection of a NBC solution on $(0,\pi)$ leads to periodic boundary conditions (PBC) on $(-\pi,\pi)$. (b) The relation between the physical domain $\Omega$ and the extended domain $\tilde{\Omega}$.

It is crucial to appreciate that when the mixed modes transform under $\gamma_1$ and $\gamma_2$ as described above, this transition is relatively unremarkable for odd parity but quite surprising for even parity [13]. With odd parity the mixed modes are symmetry related, $\gamma_1\cdot\psi^{\pm}_{23}=\psi^{\mp}_{23}$, and therefore have the same eigenvalue; hence they have the same dynamics (e.g., frequency and growth rate). This degeneracy implies that linear combinations of $\psi^{\pm}_{23}$ are also eigenstates with the same eigenvalue; which of these superpositions can develop into standing-wave patterns depends of course on nonlinear effects not included in Eq. (3). For eigenstates transforming as $\psi^{\pm}_{23}$ above, the simultaneous existence of both nonlinear pure modes and nonlinear mixed modes is generically expected, however, and bifurcation diagram (a) in Fig. 2 is well known [5].

By contrast the even-parity mixed modes $\psi^{\pm}_{31}$ are not interchanged by reflection in $x$: $\gamma_1\cdot\psi^{\pm}_{31}=-\psi^{\pm}_{31}$; consequently they are *not* related by the geometric symmetry $\Gamma$ (an observation made originally by SG). In light of the discussion for odd parity, one immediately expects (i) that $\psi^{+}_{31}$ and $\psi^{-}_{31}$ have different eigenvalues and different dynamics; (ii) that the pure modes will no longer be

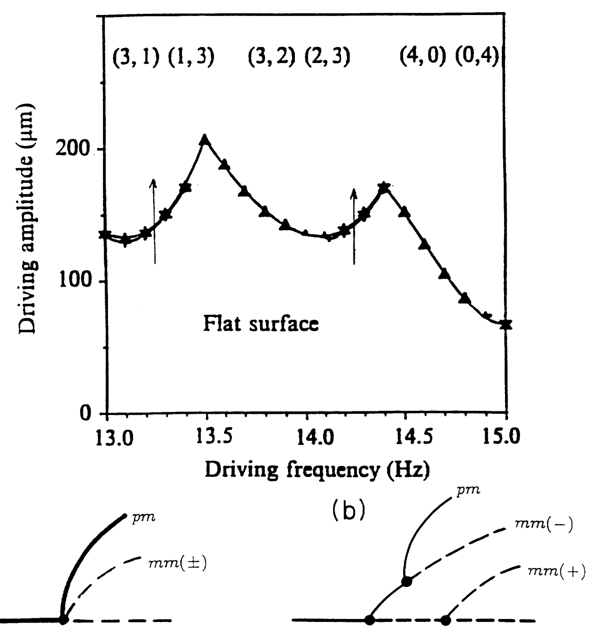


FIG. 2. Parameter space for the Faraday experiment in a square container showing the stability boundaries for the onset of standing waves with various mode numbers $(m,n)$. The bifurcation observed along the paths marked by the vertical arrows is represented in diagram (a). Solid branches are stable; the branch *pm* represents both pure modes, $\psi_{mn}$ and $\psi_{nm}$, and he branch $mm(\pm)$ represents the nonlinear extension of $\psi^{\pm}_{mn}$. Diagram (b) indicates the effect on the even-parity transition of deforming the cross section as in Fig. 3.

eigenstates and consequently will not appear as primary branches in the bifurcation diagram, and (iii) that the initial standing-wave pattern will have diagonal reflection symmetry (even or odd depending on whether $\psi_{31}^{+}$ or $\psi_{31}^{-}$ becomes unstable first). However, all of these expectations appear to fail in the experiment. Moreover (i) and (ii) also fail in the linear theory of Benjamin and Ursell: $\psi_{31}^{\pm}$ have the same eigenvalue and the pure modes are eigenfunctions.

It is natural to suspect that $\psi_{31}^{\pm}$ are somehow symmetry related after all, and this connection was recently established for the model of an ideal fluid layer obeying NBC as in Eq. (1) [13]. The idea is illustrated in Fig. 1: With NBC, a standing-wave pattern can be smoothly extended by reflection in $x$ and $y$ to give a solution to the corresponding free-boundary problem posed on the larger domain $\tilde{\Omega}$ with *periodic* boundary conditions (PBC). The geometric symmetry $\tilde{\Gamma}$ of $\tilde{\Omega}$ is generated by $\tilde{\gamma}_1$ (reflection in $x$ about $x=0$), $\gamma_2$ as before, *and* translations in $x$ and $y$ (due to PBC). Thus we can solve the original problem by calculating the bifurcation of standing waves posed on $\tilde{\Omega}$ with PBC and then restricting to those solutions that also satisfy NBC on $\Omega$. The benefit of this viewpoint is it allows one to recognize special features in the original problem that are inherited from the larger symmetry $\tilde{\Gamma}$ on $\tilde{\Omega}$ [14]. For example, the eigenstates (2) on $\tilde{\Omega}$ have the form $\tilde{\psi}_{nm}=\exp[i(mx+ny)]$ and the physical mixed modes $\psi_{mn}^{\pm}$ can be written as linear combinations of $\tilde{\psi}_{mn}$ and $\tilde{\Gamma}$-related states by reexpressing the Fourier expansion of $\psi_{mn}^{\pm}$:

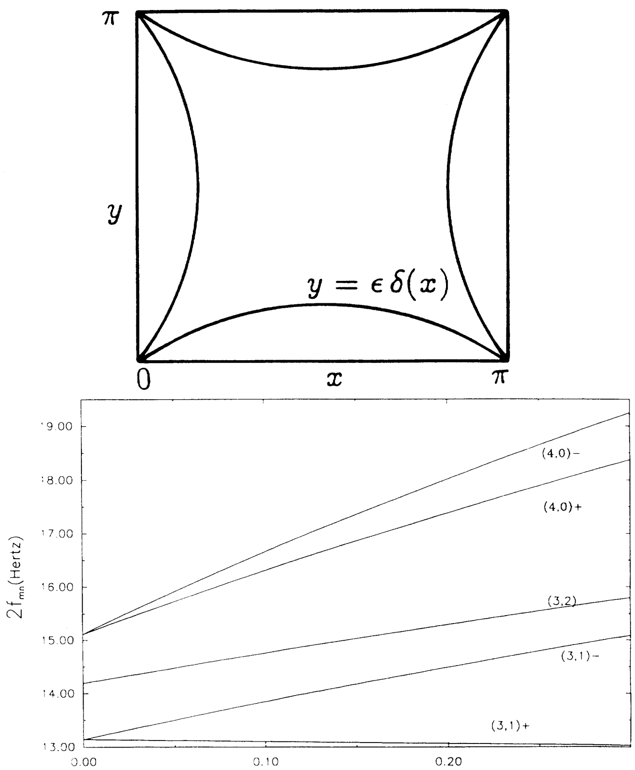


FIG. 3. Top: The deformation $y=\epsilon\delta(x)$ yields a nonsquare cross section with square symmetry. Bottom: The frequencies $f_{mn}$ of the modes in Fig. 1 for $\delta(x)=\sin x$. Mixed-mode frequencies $f_{mn}^{\pm}$ are denoted by $(m,n)\pm$.

$$\psi_{mn}^{\pm}=\frac{1}{2\pi}[\tilde{\psi}_{mn}+\gamma_2\tilde{\gamma}_1\gamma_2\cdot\tilde{\psi}_{mn}+\gamma_2\tilde{\gamma}_1\gamma_2\tilde{\gamma}_1\cdot\tilde{\psi}_{mn}+\tilde{\gamma}_1\cdot\tilde{\psi}_{mn}\pm(\gamma_2\cdot\tilde{\psi}_{mn}+\gamma_2\tilde{\gamma}_1\cdot\tilde{\psi}_{mn}+\tilde{\gamma}_1\gamma_2\tilde{\gamma}_1\cdot\tilde{\psi}_{mn}+\tilde{\gamma}_1\gamma_2\cdot\tilde{\psi}_{mn})]\,. \tag{5}$$

Here each term in Eq. (5) represents an eigenstate for the same eigenvalue $\kappa_{mn}$ and hence $\psi_{mn}^{\pm}$ must have the same eigenvalue; this explains the degeneracy of the mixed modes with even parity. The implications of the larger symmetry $\tilde{\Gamma}$ for the nonlinear behavior of waves satisfying NBC on $\Omega$ can be studied using appropriate normal forms; one finds that indeed the simultaneous appearance of pure and mixed-mode branches is expected as described by Fig. 2 [13].

The extension from $\Gamma$ to $\tilde{\Gamma}$ requires NBC on an initial cross section $\Omega$ such that reflection in $x$ and $y$ yields PBC on the extended domain. The theory can be tested by deforming $\Omega$ to a cross section $\Omega'$ which lacks this property while retaining the geometric symmetry of $\Omega$. Thus we are led to consider cross sections $\Omega'$ having nonsquare boundaries but square symmetry as illustrated in Fig. 3.

The effects of such a deformation on $\psi_{mn}^{\pm}$ are easily calculated in the model of Benjamin and Ursell by noting that the geometry of the cross section enters the linear frequency $\omega_{mn}$ only though the eigenvalue $\kappa_{mn}$ in Eq. (2).

Given the effect of the deformation on $\kappa_{mn}$ one can describe the perturbed patterns and shifted frequencies for the standing waves. Denote the deformation of $\Omega$ by $y=\epsilon\delta(x)$ as indicated in Fig. 3; the eigenfunctions $\Psi_{mn}(x,y)$ on $\Omega'$ satisfy $\nabla^2\Psi_{mn}+k_{mn}^2\Psi_{mn}=0$ subject to $\hat{\mathbf{n}}\cdot\nabla\Psi_{mn}|_{\partial\Omega'}=0$, where $\partial\Omega'$ denotes the boundary of $\Omega'$. By introducing the Green function for the unperturbed problem we write an integral equation [15] for $\Psi_{mn}$,

$$\Psi_{mn}(\mathbf{x}')=\sum_{pq}\frac{\psi_{pq}(\mathbf{x}')}{k_{mn}^2-\kappa_{pq}^2}\int_{\partial\Omega'}ds\,\Psi_{pq}(x(s))\hat{\mathbf{n}}\cdot\nabla_{\perp}\psi_{pq}(\mathbf{x}(s))\,, \tag{6}$$

and solve for $\Psi_{mn}$ and $k_{mn}^2$ iteratively using the initial choice

$$\Psi_{mn}^{(0)}=a\psi_{mn}+b\psi_{nm}\,. \tag{7}$$

Here $a$ and $b$ represent the amplitudes of the unperturbed states in an *a priori* arbitrary linear combination satisfying the normalization $a^2+b^2=1$. Inserting Eq. (7) into the right-hand side of (6) yields $\Psi_{mn}^{(1)}$ in the presence of the perturbation:

$$\Psi_{mn}^{(1)}=(k_{mn}^2-\kappa_{mn}^2)^{-1}[(aA_{mnmn}+bA_{nmmn})\psi_{mn}+(aA_{mnnm}+bA_{nmnm})\psi_{nm}]+\sum_{pq}\frac{\psi_{pq}}{k_{mn}^2-\kappa_{pq}^2}[aA_{mnpq}+bA_{nmpq}]\,, \tag{8}$$

where the sum on $p,q$ in Eq. (8) omits $(m,n)$ and $(n,m)$. The overlap integral (around the perturbed boundary),

$$A_{mnpq} \equiv \int_{\partial\Omega'} ds\, \psi_{mn}(\mathbf{x}(s))\hat{\mathbf{n}}\cdot\nabla\psi_{pq}(\mathbf{x}(s)) \,, \tag{9}$$

vanishes when $\epsilon=0$ and can be shown to satisfy two identities: (i) For arbitrary indices $mnpq$, $A_{mnpq}\equiv A_{nmqp}$, and (ii) when either $m+p$ or $n+q$ is odd, $A_{mnpq}\equiv 0$. Consistency between (7) and (8) now requires

$$\begin{aligned} a &= \lim_{\epsilon\to 0}\left[\frac{aA_{mnmn}+bA_{mnnm}}{k_{mn}^2-\kappa_{mn}^2}\right], \\ b &= \lim_{\epsilon\to 0}\left[\frac{aA_{mnnm}+bA_{mnmn}}{k_{mn}^2-\kappa_{mn}^2}\right]. \end{aligned} \tag{10}$$

Forming the ratio of these two equations and cross multiplying yields

$$(a^2-b^2)\frac{\partial A_{mnnm}}{\partial\epsilon}\bigg|_{\epsilon=0}=0\,; \tag{11}$$

this constraint is identically satisfied for odd parity when $A_{mnnm}\equiv 0$, but for $m+n$ even, we must take $a^2=b^2$ or $b=\pm a$ and $a=1/\sqrt{2}$. Thus the mixed modes $\psi_{mn}^{\pm}$ are the correct basis set to treat the perturbation for even parity. After setting $a=1/\sqrt{2}$ and $b=\pm 1/\sqrt{2}$, the perturbed eigenstates can be written as

$$\begin{aligned} \Psi_{mn}^{\pm} &= \frac{1}{\sqrt{2}}[1+O(\epsilon)]\psi_{mn}^{\pm} \\ &\quad + \frac{1}{\sqrt{2}}\sum_{pq}\frac{A_{mnpq}\psi_{pq}^{\pm}}{(k_{mn}^{\pm})^2-\kappa_{pq}^2}+O(\epsilon^2)\,. \end{aligned} \tag{12}$$

The eigenvalues for the perturbed mixed modes follow similarly from Eq. (10),

$$\begin{aligned} (k_{mn}^{\pm})^2 &= [\kappa_{mn}^2+A_{mnmn}+O(\epsilon^2)] \\ &\quad \pm [A_{mnnm}+O(\epsilon^2)]\,, \end{aligned} \tag{13}$$

and the perturbed wave frequencies are given by $(\omega_{mn}^{\pm})^2=k_{mn}^{\pm}\tanh(k_{mn}^{\pm}h)[g+\delta(k_{mn}^{\pm})^2/\rho]$. As expected from the symmetry analysis, the wave frequencies split only if the parity is even since the deformation breaks the $\tilde{\Gamma}$ symmetry but not the square symmetry; this splitting should be observable. For parameters of Ref. [10] and the particular deformation $\delta(x)=\sin x$, the frequency variation with $\epsilon$ based on Eq. (13) is shown in Fig. 3. Note also that the perturbed wave patterns in Eq. (12) still correspond to mixed modes in the sense that the diagonal reflection symmetry is retained, $\gamma_2\cdot\Psi_{mn}^{\pm}=\pm\Psi_{mn}^{\pm}$, although there are now many wave numbers present in the pattern.

The variation of the nonlinear behavior with $\epsilon$ is not fully understood especially in the neighborhood of the mode interactions (i.e., points of multiple linear instability). For the specific bifurcation diagram in Fig. 2(a), the splitting of the linear eigenvalues for the mixed modes should lead to the perturbed diagram of Fig. 2(b). In this diagram the mixed modes now branch independently from the basic state. The reflection symmetry of the stable mixed mode is subsequently broken in a secondary pitchfork bifurcation yielding states which at large amplitude (and small $\epsilon$) resemble the pure modes of the unperturbed system.

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