Complex Excitations in the Thermodynamic Bethe-Ansatz Approach

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(Received 21 March 1991)

We relate the complex solutions of the thermodynamic Bethe-ansatz equations to the excited states in factorizable massive models. This approach permits one to recover exactly the ultraviolet behavior of these excitations. In order to illustrate our method we consider several unitary and nonunitary integrable models. We also comment on the calculation of critical exponents in massless integrable models.

PACS numbers: 05.50.+q, 64.60.Cn, 75.10.Hk, 75.10.Jm

Conformal invariance [1] and integrable models [2] have been important subjects in $(1+1)$-dimensional quantum field theory. Recently, a new insight has been added. It has been argued by A. Zamolodchikov that particular perturbations around conformally invariant models lead to integrable theories [3,4]. The massive spectrum and the respective S matrices can be predicted by using symmetries, conserved charges, and bootstrap equations [4]. The relevance of these S matrices needs to be verified by studying, for example, the ultraviolet limit of the respective massive theory and then making the connection with the initial conformally invariant model. The Green's functions can be constructed in principle by solving Watson's equations [5] using the known S matrices, and the ultraviolet limit may then be taken. Unfortunately, this program has proved too cumbersome, and up to now only the scaling Ising model has been solved through this approach [6]. However, Al. Zamolodchikov has pointed out that a relativistic version of the thermodynamic Bethe ansatz [7,8] permits one to study the ground-state energy for finite temperatures. The Casimir effect for the free energy is then related to the effective central charge of the corresponding conformal field theory [9]. We will henceforth refer to this method as the TBA approach. The extension of the TBA approach to excited states remains, however, a challenging open problem.

The purpose of this Letter is to show that the TBA approach also contains information about the lowest excited states [10] of massive theories that possess a symmetry. Fortunately, there are several models that have this property including, for example, tricritical Ising, three-state Potts, tricritical three-state Potts models, and parafermionic field theories. We will consider these models and some nonunitary theories as examples of our approach. We will show that certain excited states are related to complex solutions of the TBA equations with corresponding real free energy. In the usual approach complex solutions are hypothesized in the Bethe-ansatz framework [11], and are tested by explicit diagonalization of the associated Hamiltonian or transfer matrix [12]. By contrast, although we have not recovered all the conformal scaling dimensions, in our approach the complex solutions (excitations) are obtained by studying the high-temperature TBA equations, without making any hypotheses. We also should mention that our approach will be important in theories characterized by the same central charges, but with different modular partition functions [13].

The TBA equations are usually obtained in two steps. First, one writes the Bethe-ansatz equations for the respective factorizable S matrices. The associated entropy is then written in terms of the densities of levels and particles, and the thermodynamics is encoded by minimizing the free energy [8]. For a typical theory with N particles and diagonal S matrices, the TBA equations and the respective energy $[E(R)]$ at temperature $1/R$ are given by

$$\epsilon_i(\theta)+\frac{1}{2\pi}\sum_{j=1}^{N}\int_{-\infty}^{\infty}d\theta'\,\psi_{ij}(\theta-\theta')L_j(\epsilon_j) = m_iR\cosh(\theta)\,,\quad i=1,2,\ldots,N\,, \tag{1}$$

$$E(R)=\frac{2\pi}{R}F,\qquad F=-\frac{1}{4\pi^2}\sum_{i=1}^{N}m_iR\int_{-\infty}^{\infty}d\theta\cosh(\theta)L_i(\epsilon_i)\,, \tag{2}$$

where $L_i(\epsilon_i)=\ln(1+e^{-\epsilon(\theta)})$, m_i is the mass of particle i, and ψ_{ij} is a function given in terms of the two-particle scattering amplitude S_{ij} by $\psi_{ij}=-i\,d\ln S_{ij}(\theta)/d\theta$. In Eq. (2) we wrote the function $E(R)$ in a convenient way for taking the ultraviolet limit.

We start by studying the tricritical Ising model ($c=\frac{7}{10}$) perturbed by the subleading energy perturbation. Although the associated S matrices are not diagonal [14], the TBA equations have the same form as Eq. (1), where $m_1=m$, $m_2=0$, $\psi_{11}=\psi_{22}=0$, $\psi_{12}=\psi_{21}=1/\cosh(\theta)$ [15]. In the ultraviolet limit, $R\to 0$, the variables $x=\exp(\epsilon)$, $\epsilon=\epsilon_1=\epsilon_2$, tend to constants satisfying the quadratic equation $x^2+x-1=0$. The first solution $x_1=(-1+\sqrt{5})/2$ gives the ground-state behavior, $E(R)\simeq(2\pi/R)(-7/10)$. The second solution $x_2=-(1+\sqrt{5})/2$ is imaginary in the ϵ variables, and appears as a solution of Eq. (1) with an additional complex $i\pi$ term

$$\epsilon_1+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\theta'\frac{\ln(1+e^{-\epsilon_2(\theta')})}{\cosh(\theta-\theta')}+i\pi=mR\cosh(\theta)\,,\qquad \epsilon_2+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\theta'\frac{\ln(1+e^{-\epsilon_1(\theta')})}{\cosh(\theta-\theta')}+i\pi=0\,. \tag{3}$$

The ultraviolet limit of Eq. (3) is delicate, but using the convenient shift $\epsilon' = \epsilon - i\pi$, this limit can be easily computed using standard calculations [7], and we find for $R \to 0$

$$F = -\int_0^{(-1+\sqrt{5})/2} df \left[\frac{\ln(1-f)}{f} + \frac{\ln(f)}{1-f}\right] - \int_1^{(-1+\sqrt{5})/2} df \left[\frac{\ln(1-f)}{f} + \frac{\ln(f)}{1-f}\right] = -2L\left[\frac{\sqrt{5}-1}{2}\right] + L(1) , \quad (4)$$

where $L(f_0)$ is the Rogers dilogarithm function [16]

$$L(f_0) = \frac{1}{4\pi^2}\int_0^{f_0} df \left[\frac{\ln(1-f)}{f} + \frac{\ln(f)}{1-f}\right] . \quad (5)$$

Using the exact values, $L((\sqrt{5}-1)/2) = -1/20$ and $L(1) = -1/12$, we see that $E(R) \simeq (2\pi/R)(3/40 - \frac{7}{10}/12)$ is in accordance with the conformal dimension $\Delta = 3/80$ for the tricritical Ising model. We note that the previous shift of variables changes the function $L_i(\epsilon(\theta))$ to $\ln(1-e^{-\epsilon_i(\theta)})$. This fact was first observed in a class of nonunitary $M_{3,3n\pm1}$ integrable models [17], motivated in the scaling Ising model. We can perform the same analysis described above for this nonunitary model, and we find that only the TBA equation of the lightest particle has a complex term. This complex solution corresponds to the conformal dimension of the operator $\phi_{1,n\pm1}$ in the Kac table of $M_{3,3n\pm1}$ (lowest excited state).

The three-state Potts model ($c = \frac{4}{5}$) perturbed by the thermal operator is a very interesting example in our approach. In this theory we have a particle and an antiparticle scattering with the S matrices given in Ref. [18]. Using these known S matrices, we have solved Eq. (1) for $R \to 0$ in terms of the variable $x = \exp(\epsilon)$. We find two solutions $x = (1 \pm \sqrt{5})/2$. The positive solution is related to the ground state. The negative solution is complex in the ϵ variable, namely, $\epsilon_1 = \epsilon_2^*$. In this case we should add a phase term $\pm 2\pi i/3$ to the left-hand side of Eq. (1). The ultraviolet limit has an extra term due to this phase, which should be added to the usual Rogers dilogarithm terms. The extra term has been evaluated to be $\frac{1}{3}$, so that we have

$$F = 2L(1) + 2\int_1^{(3+\sqrt{5})/2} df \left[\frac{\ln(f-1)}{f} + \frac{\ln(f)}{1-f}\right] + \frac{1}{3} . \quad (6)$$

Changing variables to $1 - 1/f = f'$, our final result is

$$F = 2L(1) + 2L\left[\frac{\sqrt{5}-1}{2}\right] + \frac{1}{3} = \left[\frac{2}{15} - \frac{4/5}{12}\right] , \quad (7)$$

giving the dimension $\Delta = \frac{1}{15}$, which appears in the modular invariant partition function of the three-state Potts model [13].

The next example is the tricritical three-state Potts model ($c = \frac{6}{7}$) perturbed by the thermal operator. The respective S matrices are related to the exceptional E_6 group [19]. Again, the complex solution corresponds to adding phases $\pm 2\pi i/3$ in Eq. (1). The calculation of the ultraviolet limit is now more complicated and we only present our final result,

$$F = \sum_1^6 L(f_i) + 4L(1) + \frac{2}{3} , \quad (8)$$

where

$$f_1 = f_2 = \frac{\sin(\pi/14)}{\sin(3\pi/14)} , \quad f_3 = f_4 = \frac{\sin(2\pi/7)\sin(\pi/7)}{\sin^2(3\pi/14)} ,$$

$$f_5 = \frac{\sin^4(3\pi/14)}{\sin^2(4\pi/7)\sin^2(\pi/7)} , \quad f_6 = \frac{\sin^2(3\pi/14)}{\sin^2(5\pi/14)} .$$

Using the *magic* sum rule $\sum_i^6 L(f_i) = -13/42$ we find $F = 2/21 - \frac{6}{7}/12$, in accordance with $\Delta = 1/21$ for the tricritical three-state Potts model.

Another beautiful example is the $Z(5)$ parafermionic field theory [18,20]. In this model we find two complex solutions (adding phases proportional to $\pm 2\pi i/5$) which are associated with the order-disorder fields with anomalous dimension $\Delta = 2/35, 3/35$.

We can also compare the numerical results of the TBA equations with perturbation theory by the respective conformal operator. For example, solving Eq. (3) numerically for $mR < 1$, we can compute the values of the scaling function $F(mR)$. The first term in the expansion of $F(mR)$ in powers $(mR)^{4/5}$ is proportional to the structure constant $C_{(2,2)(1,3)(2,2)}$ of the tricritical Ising model. Our numerical estimate is 0.15258, in good agreement with the exact value [21]

$$C_{(2,2)(1,3)(2,2)} = \frac{1}{6}\left[\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}\right]^{1/2} = 0.152576 \ldots .$$

Finally, we mention that the ideas discussed above can also be applied to critical integrable models. It is known that the operator content of the integrable Heisenberg spin-s chain is a composition of a free-bosonic field and a $Z(2s)$-invariant model [22]. Our results together with previous work [23] predict exactly the lowest exponents $x = \frac{3}{8}$ ($s = 1$), $x = \frac{3}{10}$ ($s = 3/2$), reproducing the known anomalous dimensions of the two-dimensional Wess-Zumino model [24].

We believe that the results of this Letter are the first steps toward the generalization of the TBA approach to excited states. A more detailed account of this work, including a discussion on the completeness of the complex solution and applications of our approach to integrable critical models, will be published elsewhere.

This work was partially supported by CNPQ (Brazilian agency) and NSF Grant No. PHY 86-14185. The author is indebted to J. L. Cardy and J. Miller for valuable comments on the manuscript.

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